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MEAN TIME TO RECRUITMENT FOR A MULTIGRADE MANPOWER SYSTEM WITH SINGLE THRESHOLD, SINGLE SOURCE OF DEPLETION WHEN WASTAGES FORM AN ORDER STATISTICS

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Abstract:
In this paper a multi graded organization in which depletion of man powers occur due to its policy decisions taken by the organization is considered. Four cases are constructed by taking exponential thresholds for the loss of man powers in each grade, where the loss of man powers (wastages) form an order statistics and inter decision times form i) an ordinary renewal process ii) an order statistics iii) a geometric process iv) correlated. Mean time to recruitment is obtained using an univariate CUM policy of recruitment (i.e) "The organization survives iff atleast $r, (1 \leq r \leq n)$ out of n-grades survives in the sense that threshold crossing has not take place in these grades". The influence of the nodal parameters on the system characteristics is studied and relevant conclusions are presented.

Key Words: Loss of Man Powers, Inter Decision Time, Order Statistics & Exponential Thresholds.

Introduction:

Exits of personal which is in other words known as wastage, is an important aspect in the study of manpower planning. Many models have been discussed using different types of wastages and also different types of distribution for the loss of man powers, the thresholds and inter decision times. Such models are seen in [1] and [2]. In [3], [4], [5] and [6] the authors have obtained the mean time to recruitment in a two grade manpower system based on order statistics by assuming different distribution for thresholds. In [8] for a two grade manpower system with two types of decisions when the wastages form a geometric process is obtained. The problem of time to recruitment is studied by several authors for the organizations consisting of single grade/two grade/ three grades .More specifically for a two grade system, in all the earlier work, the threshold for the organization is minimum or maximum or sum of the thresholds for the loss of manpower in each grades, no attempt has been made so far to design a comprehensive recruitment policy for a system with two or three grades. In [10], [11] & [12] a new design for a comprehensive univariate CUM recruitment policy of manpower system is used with n grades in order to bring results proved independently for maximum, minimum model as a special case. In all previous work, the problem of time to recruitment is studied for only an organization consisting of atmost three grades. In [11], [12] author has worked on this comprehensive univariate policy when wastages form ordinary renewal process and interdecision time form geometric and order statistics. In this paper an organization with n-grades is considered and the mean time to recruitment are obtained using an appropriate univariate CUM policy of recruitment (i.e)"The organization survives iff atleast $r, (1 \leq r \leq n)$ out of n-grades survives in the sense that threshold crossing has not take place in these grades", when wastages form an order statistics.

Model Description and Assumptions:

- ✓ An organization having two grades in which decisions are taken at random epochs in $(0, \infty)$ and at every decision making epoch a random number of persons quit the organization. There is an associated loss of man hours to the organization if a person quits.
- ✓ It is assumed that the loss of man hours is linear and cumulative.
- ✓ The loss of manpower at any decision epoch forms a sequence of independent and identically distributed random variables which form order statistics.
- ✓ The inter-decision times are independent and identically distributed random variables.
- ✓ The loss of manpower process and the process of inter-decision times are statistically independent.
- ✓ The thresholds for the n-grades are independent and identically distributed exponential random variable.
- ✓ Univariate CUM policy of recruitment: "The organization survives iff atleast $r, (1 \leq r \leq n)$ out of n-grades survives in the sense that threshold crossing has not take place in these grades"

Notations:

X_i:the continuous random variable denoting the amount of depletion caused to the organization due to the exit of persons corresponding to the j^{th} decision,j=1,2,3… and X_i's form an order statistics.
$G(x)$: Distribution function of x such that $(x) = 1 - e^{-cx}$, $g(x)$: probability density function.
$\{X_i\}$,i=1 to m be a sample of size m which forms an order statistics.
$g_{x(1)}(.), g_{x(2)}(.) \ldots . g_{x(m)}(.)$: Density function of $x(1), x(2), \ldots . x(m)$.
$G_k(.)$: The distribution function of $S_k = \sum_{i=1}^{k} X_i$
$g_k(.)$ its probability density function.
$U_i: i = 1,2,3 \ldots$ The inter decision time between $(i-1)^{th}$ and i^{th} decision.
$F(.)[f(.)]$: Distribution (density) function.

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$F_k(.), f_k(.)$: The distribution (density) function of $\sum_{i=1}^{k} U_i$. $V_k(t)$: The probability that there are exactly k decision making epoch in $(0, t]$.
N(t): the number of policy decisions.
$V_k(t)$: Probability that there are exactly k decisions taken in (0, t].
Y_j: The continuous random variable denoting the thresholds for the j^{th} grade.
Y: The continuous random variable denoting the thresholds for the organization.
$H(.)$: The distribution functions of Y.
T_j: Time taken for threshold crossing in the j^{th} grade,j=1,2,3…n.
T : Time to recruitment of the organization
$E(T)$: Mean time to recruitment.

Main Result:

The survival function of the time to recruitment is given by

$$P(T > t) = \sum_{k=0}^{\infty} P\left(\begin{array}{c} Exactly\ k\ decision\ epoch(0,t] and\ the\ threshold \\ level\ Y\ is\ not\ crossed by\ the\ total\ loss\ of\ manhours\ in\ these\ k\ decisions\ in\ atleast\ r\ grades \end{array}\right)$$

$$i.e\ \ P(T > t) = \sum_{k=0}^{\infty} V_k(t) P\left(\sum_{i=1}^{k} X_i < Y\right) \qquad (1)$$

By the law of total probability

$$P\left(\sum_{i=}^{k} X_i < Y\right) = \int_0^{\infty} \mathrm{P}\left[\mathrm{Y} > \sum_{i=1}^{k} x_i / \sum_{i=1}^{k} x_i = x\right] g_k(x)\mathrm{d}x \quad = \int_0^{\infty} g_k(x)\,[1 - H(x)]dx.$$

$$= \int_0^{\infty} g_k(x) \sum_{i=r}^{n} nC_i\,[1 - H(x)]^i [H(x)]^{n-i} dx.$$

$$= \int_0^{\infty} g_k(x) \sum_{i=r}^{n} nC_i \left[e^{-\theta x}\right]^i \left[1 - e^{-\theta x}\right]^{n-i} dx$$

$$= \sum_{i=r}^{n} nC_i \int_0^{\infty} g_k(x) e^{-i\theta x} \left[1 - e^{-\theta x}\right]^{n-i} dx \qquad (2)$$

Using binomial expansion

$$= \sum_{i=r}^{n} nC_i \int_0^{\infty} g_k(x) e^{-i\theta x} \left[1 - (n-i)C_1 e^{-\theta x} + (n-i)C_2 e^{-(i+2)\theta x} + \ \ldots + (-1)^{n-i} e^{-n\theta x}\right] dx.$$

$$= \sum_{i=r}^{n} nC_i \left[\overline{g_k}(i\theta) - (n-i)C_1 \overline{g_k}((i+1)\theta) + (n-i)C_2 \overline{g_k}((i+2)\theta) + \ \ldots (-1)^{n-i} \overline{g_k}(n\theta)\right] \qquad (3)$$

From renewal theory $V_k(t) = F_k(t) - F_{k+1}(t)$ with $F_0(t) = 1$ (4)

Substituting (3) and (4) in (1) we get,

$$P(T > t) = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] \sum_{i=r}^{n} nC_i \left[\overline{g_k}(i\theta) - (n-i)C_1 \overline{g_k}((i+1)\theta) + (n-i)C_2 \overline{g_k}((i+2)\theta) + \ \ldots (-1)^{n-i} \overline{g_k}(n\theta)\right] \ (5)$$

$$P(T > t) = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] \sum_{i=r}^{n} nC_i \left[[\bar{g}(i\theta)]^k - (n-i)C_1 [\bar{g}((i+1)\theta)]^k + (n-i)C_2 [\bar{g}((i+2)\theta)]^k + \ \ldots (-1)^{n-i} [\bar{g}(n\theta)]^k\right]$$

The probability function $x(1) and\ x(m)$ are given by (sheldonRoss 2005)

$$g_{x(j)}(x) = j \binom{m}{j} [G(x)]^{j-i} g(x) [1 - G(x)]^{m-j}, j = 1,2,3 \ldots m \qquad (6)$$

Therefore the probability density function of $x(1) and\ x(m)$ are given by

$$g_{x(1)}(x) = m\, g(x) [1 - G(x)]^{m-1} \qquad (7)$$

$$g_{x_{(m)}}(t) = m\, g(x) [G(x)]^{m-1} \qquad (8)$$

We shall now obtain the mean time to recruitment according as $g(x) = g_{x_{(1)}}(x)$ or $g(x) = g_{x_{(m)}}(x)$

Suppose $g(x) = g_{x_{(1)}}(x)$

Then

$$P(T > t) = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] \sum_{i=r}^{n} nC_i \left[\left[\bar{g}_{x(1)}(i\theta)\right]^k - (n-i)C_1 \left[\bar{g}_{x(1)}((i+1)\theta)\right]^k + (n-i)C_2 \left[\bar{g}_{x(1)}((i+2)\theta)\right]^k + \ldots -1 n-i [gx(1)(n\theta)]k \right] \qquad (9)$$

$$L(t) = 1 - P(T > t)$$

$$l(t) = \frac{d}{dt}(L(t)) = -\sum_{k=0}^{\infty} [f_k(t) - f_{k+1}(t)] \sum_{i=r}^{n} nC_i \left[\left[\bar{g}_{x(1)}(i\theta)\right]^k - (n-i)C_1 \left[\bar{g}_{x(1)}((i+1)\theta)\right]^k + (n-i)C_2 \left[\bar{g}_{x(1)}((i+2\theta)\right)k + \ldots -1 n-i [gx(1)(n\theta)]k \right] \qquad (10)$$

$$\bar{l}(s) = \sum_{i=r}^{n} nC_i \left\{ \left[1 - \bar{g}_{x(1)}(i\theta)\right] \sum_{k=0}^{\infty} \bar{f_k}(s) \left[\bar{g}_{x(1)}(i\theta)\right]^{k-1} - (n-i)C_1 \left[1 - \bar{g}_{x(1)}((i+1)\theta)\right] \sum_{k=0}^{\infty} \bar{f_k}(s) \left[\bar{g}_{x(1)}((i+1)\theta)\right]^{k-1} + \ldots 1 - gx(1) n\theta k=0\infty fks gx(1) n\theta k-1 \right\} \qquad (11)$$

$$E(T) = -\frac{d}{ds}\left(\bar{l}(s)\right)_{s=0} =$$

$$\sum_{i=r}^{n} nC_i \left\{ \frac{\left[1-\bar{g}_{x(1)}(i\theta)\right]\left[\frac{d}{ds}(\bar{f}(s)\right]_{s=0}}{\left[1-\bar{g}_{x(1)}(i\theta)\bar{f}(s)\right]_{s=0}^2} - (n-i)C_1 \frac{\left[1-\bar{g}_{x(1)}((i+1)\theta)\right]\left[\frac{d}{ds}(\bar{f}(s)\right]_{s=0}}{\left[1-\bar{g}_{x(1)}((i+1)\theta)\bar{f}(s)\right]_{s=0}^2} + \cdots (-1)^{n-i} \frac{\left[1-\bar{g}_{x(1)}(n\theta)\right]\left[\frac{d}{ds}(\bar{f}(s)\right]_{s=0}}{\left[1-\bar{g}_{x(1)}(n\theta)\bar{f}(s)\right]_{s=0}^2} \right\} \qquad (12)$$

$$= -\sum_{i=r}^{n} nC_i \left\{ \frac{1}{\left[1-\bar{g}_{x(1)}(i\theta)\right]} - (n-i)C_1 \frac{1}{\left[1-\bar{g}_{x(1)}((i+1)\theta)\right]} + \cdots (-1)^{n-i} \frac{1}{\left[1-\bar{g}_{x(1)}(n\theta)\right]} \right\} \left[\frac{d}{ds}(\bar{f}(s)\right]_{s=0} \qquad (13)$$

Since $g(x) = ce^{-cx}$

$$g_{x(1)}(x) = mce^{-cx}(e^{-cx})^{m-1} = mce^{-mcx}$$

$$\bar{g}_{x(1)}(\theta) = mc \int_0^{\infty} e^{-mcx} e^{-\theta x} dx = mc \int_0^{\infty} e^{-(mc+\theta)x} dx = mc \left[\frac{e^{-(mc+\theta)x}}{-(mc+\theta)}\right]_0^{\infty} = \frac{mc}{mc+\theta} \qquad (14)$$

We now obtain the analytical result for variance of time to recruitment in closed form for four different cases on inter-decision times $\{U_i\}_1^\infty$

Case (i) $\{U_i\}_{i=1}^\infty$ form an Ordinary Renewal Process:

The inter decision times are assumed to be independent and identically distributed hyper exponential random variable with probability density function $f(t) = pe^{-\lambda_h t} + qe^{-\lambda_l t}, p + q = 1$.Where λ_h, λ_l are high and low attrition rate ,p,q are the proportion of decisions having high and low attrition.

$$\bar{f}(s) = \frac{p\lambda_h}{\lambda_h+s} + \frac{q\lambda_l}{\lambda_l+s}, \bar{f}(0) = \frac{p\lambda_h}{\lambda_h} + \frac{q\lambda_l}{\lambda_l} = p + q = 1,$$

$$\frac{d}{ds}\left(\bar{f}(s)\right) = \frac{-p\lambda_h}{(\lambda_h+s)^2} + \frac{-q\lambda_l}{(\lambda_l+s)^2}$$

$$\left(\frac{d}{ds}\left(\bar{f}(s)\right)\right)_{s=0} = -\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right), \bar{f}(0) = 1 \quad (15)$$

Substituting (15) in equation (13), we get

$$E(T) = \left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n} nC_i \left\{\frac{1}{[1-\bar{g}_{x(1)}(i\theta)]} - (n-i)C_1 \frac{1}{[1-\bar{g}_{x(1)}((i+1)\theta)]} + \cdots (-1)^{n-i} \frac{1}{[1-\bar{g}_{x(1)}(n\theta)]}\right\} \quad (16)$$

Subtitling the (14) in (16), we get

$$E(T) = \left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n} nC_i \left\{\left(\frac{mc+i\theta}{i\theta}\right) - (n-i)C_1 \left(\frac{mc+(i+1)\theta}{(i+1)\theta}\right) \dots (-1)^{n-i}\left(\frac{mc+n\theta}{n\theta}\right)\right\} \quad (17)$$

Case (ii) $\{U_i\}_{i=1}^\infty$ form an Order Statistics:

Consider the population $\{U_i\}_{i=1}^\infty$ of independent and identically distributed interdecision times with hyper exponential cumulative distribution $F(t) = 1 - pe^{-\lambda_h t} - qe^{-\lambda_l t}$ and the corresponding density function f(t).Assume that $\{U_i\}_{i=1}^{m_1}$ be a sample of size m selected from this population .Let $U_1, U_2, \dots, U_{m_1}$ be the order statistics corresponding to this sample with respective probability density function $f_{U_1}, f_{U_2}, \dots f_{U_{m_1}}$. U_1 is the first order statistics and U_{m1} is the m_1^{th} order statistics such that $U_1 \le U_2 \le \cdots \le U_{m_1}$ and hence not independent.

The probability density function of jth order statistics is given by[Sheldon M.Ross2005]

$$f_{U_j}(t) = j\binom{m_1}{j}[F(t)]^{j-i} f(t)[1-F(t)]^{m_1-j}, j = 1,2,3 \dots m_1$$

Therefore the probability density function of U_1 and U_{m_1} are given by

$$f_{U_1}(t) = m_1 f(t)[1-F(t)]^{m_1-1}$$

$$f_{U_{m_1}}(t) = m_1 f(t)[F(t)]^{m_1-1}$$

Sub Case (i)

If $f(t) = f_{U_1}(t)$

$$\bar{f}(s) = \bar{f}_{U_1}(s) = \int_0^\infty e^{-st} m_1 f(t)[1-F(t)]^{m_1-1}dt = \int_0^\infty e^{-st}[-[d(1-F(t))]^{m_1}]$$

$$\bar{f}(s) = \int_0^\infty e^{-st}[-(d(pe^{-\lambda_h t} + qe^{-\lambda_l t})^{m_1}]$$

By using binomial expansion

$$\bar{f}(s) = \int_0^\infty e^{-st}[-d(\textstyle\sum_{r_1=0}^{m}\binom{m_1}{r_1}p^{r_1}q^{m_1-r_1}e^{-(\lambda_h r_1-\lambda_l r_1+\lambda_l m_1)t})]$$

$$= \sum_{r_1=0}^{m_1}\binom{m_1}{r_1}p^{r_1}q^{m_1-r_1}\int_0^\infty e^{-st}(\lambda_h r_1 - \lambda_l r_1 + \lambda_l m_1)e^{-(\lambda_h r_1-\lambda_l r_1+\lambda_l m_1)t}dt$$

$$= \sum_{r_1=0}^{m_1}\binom{m_1}{r_1}p^{r_1}q^{m_1-r_1}(\lambda_h r_1 - \lambda_l r_1 + \lambda_l m_1)\left[\frac{1}{(s+\lambda_h r_1-\lambda_l r_1+\lambda_l m_1)}\right]$$

$$\frac{d}{ds}\left(\bar{f}(s)\right) = \sum_{r_1=0}^{m_1}\binom{m_1}{r_1}p^{r_1}q^{m_1-r_1}(\lambda_h r_1 - \lambda_l r_1 + \lambda_l m_1)\left[\frac{-1}{(s+\lambda_h r_1-\lambda_l r_1+\lambda_l m_1)^2}\right]$$

$$\left[\frac{d}{ds}\left(\bar{f}(s)\right)\right]_{s=0} = -\sum_{r_1=0}^{m_1}\binom{m_1}{r_1}p^{r_1}q^{m_1-r_1}\left[\frac{1}{(\lambda_h r_1-\lambda_l r_1+\lambda_l m_1)}\right] \quad (18)$$

Substituting (18) in (13) we get

$$E(T) = \sum_{r_1=0}^{m_1}\binom{m_1}{r_1}p^{r_1}q^{m_1-r_1}\left[\frac{1}{(\lambda_h r_1-\lambda_l r_1+\lambda_l m_1)}\right] \times \sum_{i=r}^{n} nC_i \left\{\frac{1}{[1-\bar{g}_{x(1)}(i\theta)]} - (n-i)C_1\frac{1}{[1-\bar{g}_{x(1)}((i+1)\theta)]} + \cdots (-1)^{n-i}\frac{1}{[1-\bar{g}_{x(1)}(n\theta)]}\right\} \quad (19)$$

Substituting (14) in (19) we get

$$E(T) = \sum_{r_1=0}^{m_1}\binom{m_1}{r_1}p^{r_1}q^{m_1-r_1}\left[\frac{1}{(\lambda_h r_1-\lambda_l r_1+\lambda_l m_1)}\right] \times \sum_{i=r}^{n} nC_i \left\{\left(\frac{mc+i\theta}{i\theta}\right) - (n-i)C_1\left(\frac{mc+(i+1)\theta}{(i+1)\theta}\right) + \cdots (-1)^{n-i}\left(\frac{mc+n\theta}{n\theta}\right)\right\} \quad (20)$$

Sub case (ii)

$$f(t) = f_{U_{m_1}}(t) = m_1 f(t)[F(t)]^{m_1-1}$$

$$\bar{f}(s) = \bar{f}_{U_{m_1}}(s) = \int_0^\infty e^{-st} m_1 f(t)[F(t)]^{m_1-1}dt$$

$$= \int_0^\infty e^{-st} d(F(t))^{m_1} = \int_0^\infty e^{-st}[d(1 - pe^{-\lambda_h t} - qe^{-\lambda_l t})]^{mm_1}$$

$$= \int_0^\infty e^{-st} d(\textstyle\sum_{r_1=0}^{m_1}\binom{m_1}{r_1}(-1)^{m_1-r_1}1^{r_1}(pe^{-\lambda_h t} + qe^{-\lambda_l t})^{m_1-r_1}$$

$$\bar{f}(s) = \sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_1-r_1}(-1)^{m_1-r_1+1}\frac{m_1!}{r_1!r_2!(m_1-r_1-r_2)!}p^{r_2}q^{m_1-r_1-r_2}\frac{(\lambda_h r_2-\lambda_l r_2-\lambda_l r_1+\lambda_l m_1)}{(s+\lambda_h r_2-\lambda_l r_2-\lambda_l r_1+\lambda_l m_1)}$$

$$\left[\frac{d}{ds}\left(\bar{f}(s)\right)\right]_{s=0} = -\sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_1-r_1}(-1)^{m_1-r_1+1}\frac{m_1!}{r_1!r_2!(m_1-r_1-r_2)!}p^{r_2}q^{m_1-r_1-r_2}\frac{1}{(\lambda_h r_2-\lambda_l r_2-\lambda_l r_1+\lambda_l m_1)} \quad (21)$$

Substituting (21) in (13) the mean time to recruitment is

$$E(T)=\sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_1-r_1}\frac{m_1!}{r_1!r_2!(m_1-r_1-r_2)!}p^{r_2}q^{m_1-r_1-r_2}\frac{1}{(\lambda_h r_2-\lambda_l r_2-\lambda_l r_1+\lambda_l m_1)}\times\sum_{i=r}^{n}nC_i\left\{\frac{1}{[1-\bar{g}_{x(1)}(i\theta)]}-(n-i)C_1\frac{1}{[1-\bar{g}_{x(1)}((i+1)\theta)]}+\right.$$

... (−1)n−i11−gx(1)nθ

(22)

Substituting (14) in (22), we get

$$E(T)=\sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_1-r_1}(-1)^{m_1-r_1+1}\frac{m_1!}{r_1!r_2!(m_1-r_1-r_2)!}p^{r_2}q^{m_1-r_1-r_2}\frac{1}{(\lambda_h r_2-\lambda_l r_2-\lambda_l r_1+\lambda_l m_1)}\times\sum_{i=r}^{n}nC_i\left\{\left(\frac{mc+i\theta}{i\theta}\right)-(n-i)C_1\left(\frac{mc+(i+1)\theta}{(i+1)\theta}\right)+\right.$$

...−1n−imc+nθnθ (23)

Case (iii) $\{U_i\}_{i=1}^{\infty}$ form an geometric process

Assume that the inter decision times $U_i, i=1,2,3\ldots$ form a geometric process with rate b, (b>0).It is assumed that the probability density function ofU_1 is hyper exponential density function $f(t)=p\lambda_h e^{-\lambda_h}+q\lambda_l e^{-\lambda_l}, p+q=1$.

$$\left.\begin{aligned}\bar{f}(s)&=\frac{p\lambda_h}{\lambda_h+s}+\frac{q\lambda_l}{\lambda_l+s}\\ \bar{f}(0)&=\frac{p\lambda_h}{\lambda_h}+\frac{q\lambda_l}{\lambda_l}=p+q=1\\ \frac{d}{ds}\left(\bar{f}(s)\right)&=\frac{-p\lambda_h}{(\lambda_h+s)^2}+\frac{-q\lambda_l}{(\lambda_l+s)^2}\\ \left(\frac{d}{ds}\left(\bar{f}(s)\right)\right)_{s=0}&=-\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\end{aligned}\right\}\quad(24)$$

$$\bar{f}_k(s)=\prod_{i=1}^{k}\bar{f}\left(\frac{s}{b^{i-1}}\right)$$

$$\frac{d}{ds}\left(\bar{f}_k(s)\right)=\frac{d}{ds}\left(\prod_{i=1}^{k}\bar{f}\left(\frac{s}{b^{i-1}}\right)\right)=\frac{d}{ds}\left(\bar{f}(s)\times\bar{f}\left(\frac{s}{b}\right)\times\bar{f}\left(\frac{s}{b^2}\right)\times\ldots\times\bar{f}\left(\frac{s}{b^{k-1}}\right)\right)$$

$$=\frac{d}{ds}\left(\bar{f}(s)\right)\times\prod_{i=2}^{k}\bar{f}\left(\frac{s}{b^{i-1}}\right)+\frac{1}{b}\frac{d}{ds}\left(\bar{f}(s)\right)\times\prod_{\substack{i=1\\ i\neq 2}}^{k}\bar{f}\left(\frac{s}{b^{i-1}}\right)+\frac{1}{b^2}\frac{d}{ds}\left(\bar{f}(s)\right)\times\prod_{\substack{i=1\\ i\neq 3}}^{k}\bar{f}\left(\frac{s}{b^{i-1}}\right)+\cdots+\frac{1}{b^{k-1}}\frac{d}{ds}\left(\bar{f}(s)\right)\times\prod_{i=}^{k-1}\bar{f}\left(\frac{s}{b^{i-1}}\right)$$

Using (24), we get

$$\left(\frac{d}{ds}\left(\bar{f}_k(s)\right)\right)_{s=0}=\left(\frac{d}{ds}\left(\prod_{i=1}^{k}\bar{f}\left(\frac{s}{b^{i-1}}\right)\right)\right)_{s=0}=-\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=1}^{k}\frac{1}{b^{i-1}}\quad(25)$$

Consider$\sum_{k=0}^{\infty}\left[\left[\frac{d}{ds}\left(\bar{f}_k(s)\right)\right]_{s=0}-\left[\frac{d}{ds}\left(\bar{f}_{k+1}(s)\right)\right]_{s=0}\right]=\sum_{k=0}^{\infty}-\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\left[\sum_{i=1}^{k}\frac{1}{b^{i-1}}-\sum_{i=1}^{k+1}\frac{1}{b^{i-1}}\right]$

$$=-\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{k=0}^{\infty}\frac{1}{b^k}\quad(26)$$

From equation (11), we have

$\bar{l}(s)=$

$-\sum_{k=0}^{\infty}\left[\bar{f}_k(s)-\bar{f}_{k+1}(s)\right]\times$

i=rnnCigx(1)iθk−n−iC1gx(1) i+1θk+n−iC2gx(1)(i+2θ)k …−1n−igx(1)(nθ)k

$$E(T)=-\left[\frac{d}{ds}\left(\bar{l}(s)\right)\right]_{s=0}$$

$$=\sum_{k=0}^{\infty}\left[\left[\frac{d}{ds}\left(\bar{f}_k(s)\right)\right]_{s=0}-\left[\frac{d}{ds}\left(\bar{f}_{k+1}(s)\right)\right]_{s=0}\right]\times\sum_{i=r}^{n}nC_i\left[\left[\bar{g}_{x(1)}(i\theta)\right]^k-(n-i)C_1\left[\bar{g}_{x(1)}((i+1)\theta)\right]^k+(n-i)C_2\left[\bar{g}_{x(1)}((i+\right.\right.$$

2θk …−1n−igx1nθk (27)

Substituting (26) in (27)

$$=\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{k=0}^{\infty}\frac{1}{b^k}\times\sum_{i=r}^{n}nC_i\left[\left[\bar{g}_{x(1)}(i\theta)\right]^k-(n-i)C_1\left[\bar{g}_{x(1)}((i+1)\theta)\right]^k+(n-i)C_2\left[\bar{g}_{x(1)}((i+2)\theta)\right]^k+\right.$$

…−1n−igx(1)(nθ)k

$$=\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\times\sum_{i=r}^{n}nC_i\sum_{k=0}^{\infty}\left[\left[\frac{\bar{g}_{x(1)}(i\theta)}{b}\right]^k-(n-i)C_1\left[\frac{\bar{g}_{x(1)}((i+1)\theta)}{b}\right]^k+(n-i)C_2\left[\frac{\bar{g}_{x(1)}((i+2)\theta)}{b}\right]^k\ldots(-1)^{n-i}\left[\frac{\bar{g}_{x(1)}(n\theta)}{b}\right]^k\right]$$

$$=\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n}nC_i\left[\left[1-\frac{\bar{g}_{x(1)}(i\theta)}{b}\right]^{-1}-(n-i)C_1\left[1-\frac{\bar{g}_{x(1)}((i+1)\theta)}{b}\right]^{-1}+(n-i)C_2\left[1-\frac{\bar{g}_{x(1)}((i+2)\theta)}{b}\right]^{-1}+\ldots(-1)^{n-i}\left[1-\right.\right.$$

gx(1)(nθ)b−1

$$=\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n}nC_i\left[\frac{b}{b-\bar{g}_{x(1)}(i\theta)}\right]-(n-i)C_1\left[\frac{b}{b-\bar{g}_{x(1)}((i+1)\theta)}\right]+(n-i)C_2\left[\frac{b}{b-\bar{g}_{x(1)}((i+2)\theta)}\right]+\cdots(-1)^{n-i}\left[\frac{b}{b-\bar{g}_{x(1)}((n\theta)}\right]\quad(28)$$

Using (14) in (28)

$$E(T)=\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n}nC_i\left[\frac{b(mc+i\theta)}{mc(b-1)+bi\theta}-(n-i)C_1\frac{b(mc+(i+1)\theta)}{mc(b-1)+b(i+1)\theta}+(n-i)C_2\frac{b(mc+(i+2)\theta)}{mc(b-1)+b(i+2)\theta}+\cdots(-1)^{n-i}\frac{b(mc+n\theta)}{mc(b-1)+bn\theta}\right]\quad(29)$$

Case (iv) When $U_i's$ are Correlated:

The inter decision times are assumed to be exchangeable and constantly correlated exponential random variables with mean $\frac{1}{\mu}(\mu > 0)$.Let R be the constant correlation between U_i and $U_j, i \neq j$.

By taking Laplace Stieljes transform both side using (9)

$\bar{L}(s) =$

$$-\sum_{k=1}^{\infty}[\bar{F}_k(s) - \bar{F}_{k+1}(s)] \times \sum_{i=r}^{n} nC_i[[[\bar{g}(i\theta)]^k - (n-i)C_1[\bar{g}((i+1)\theta)]^k + (n-i)C_2[\bar{g}((i+2)\theta)]^k + \ \ldots (-1)^{n-i}[\bar{g}(n\theta)]^k]$$

$$E(T) = -\left[\frac{d}{ds}(\bar{L}(s))\right]_{s=0} = \sum_{k=0}^{\infty}\left[\left[\frac{d}{ds}(\bar{F}_k(s))\right]_{s=0} - \left[\frac{d}{ds}(\bar{F}_{k+1}(s))\right]_{s=0}\right] \times \sum_{i=r}^{n} nC_i\left[\left[\bar{g}_{x(1)}(i\theta)\right]^k - (n-i)C_1\left[\bar{g}_{x(1)}((i+1)\theta)\right]^k +\right.$$

n−iC2𝑔𝑥(1)i+2θk …−1n−i𝑔𝑥(1)nθk (30)

The cumulative distribution function of the partial sum $U_1 + U_2 + \cdots + U_k$ is given by Gurland (1955) as

$$F_k(t) = \left(\frac{1-R}{1-R+kR}\right)\sum_{j=0}^{\infty}\left(\frac{kR}{1-R+kR}\right)^j \frac{\varphi(k+j,\frac{t}{u})}{(k+j-1)!}$$

$$where\ u = \frac{1-R}{\mu}\ and\ \varphi\left(k+j,\frac{t}{u}\right) = \int_0^{\frac{t}{u}} e^{-\epsilon}\, \epsilon^{k+j-1}\ d\,\epsilon.$$

$$\bar{F}_k(s) = \left(\frac{1-R}{1-R+kR}\right)\sum_{j=0}^{\infty}\left(\frac{kR}{1-R+kR}\right)^j \frac{1}{(k+j-1)!}\int_0^{\infty}\varphi(k+j,\frac{t}{u})e^{-st}\,dt$$

$$= \left(\frac{1-R}{1-R+kR}\right)\sum_{j=0}^{\infty}\left(\frac{kR}{1-R+kR}\right)^j \frac{1}{(k+j-1)!}\int_0^{\infty} e^{-st}\frac{d}{dt}\left(\int_0^{\frac{t}{u}} e^{-\epsilon}\,\epsilon^{k+j-1}\ d\,\epsilon.\right)$$

$$= \frac{1}{(1+us)^k}\left[1 + \frac{kRus}{(1-R)(1+us)}\right]^{-1}$$

$$= \left[(1+us)^k\left[1 + \frac{kRus}{(1-R)(1+us)}\right]\right]^{-1}$$

$$\bar{F}_k(s) = \frac{(1-R)m^k}{1-R+kR-kRm}\ where\ m = \frac{1}{1+us}$$

$$\frac{d}{ds}[\bar{F}_k(s)] = (1-R)\left[\frac{(1-R+kR-kRm)km^{k-1}+m^k kR}{(1-R+kR-kRm)^2}\right]\frac{d}{ds}(m)$$

$$\frac{d}{ds}(m) = -\frac{u}{(+us)^2}\ \text{and}\ \left(\frac{d}{ds}(m)\right)_{s=0} = -u\ \text{and}\ (m)_{s=0} = 1$$

$$\left[\frac{d}{ds}[\bar{F}_k(s)]\right]_{s=0} = (1-R)\left[\frac{(1-R)(-ku)-kRu}{(1-R)^2}\right] = \frac{-ku}{(1-R)}$$

$$\left[\frac{d}{ds}[\bar{F}_k(s)]\right]_{s=0} - \left[\frac{d}{ds}[\bar{F}_{k+1}(s)]\right]_{s=0} = \frac{-ku}{(1-R)} + \frac{(k+1)u}{(1-R)} = \frac{u}{(1-R)} \qquad (31)$$

Substituting (31) in (30), the mean time to recruitment is

$E(T) =$

$\sum_{k=0}^{\infty}\left[\frac{u}{1-R}\right] \times$

i=rnnCi𝑔𝑥(1)iθk−n−iC1𝑔𝑥(1) i+1θk+n−iC2𝑔𝑥(1)i+2θk+ …−1n−i𝑔𝑥(1)nθk

$$= \left[\frac{u}{1-R}\right]\sum_{i=r}^{n} nC_i\sum_{k=0}^{\infty}\left[\left[\bar{g}_{x(1)}(i\theta)\right]^k - (n-i)C_1\left[\bar{g}_{x(1)}((i+1)\theta)\right]^k + (n-i)C_2\left[\bar{g}_{x(1)}((i+2)\theta)\right]^k + \ \ldots (-1)^{n-i}\left[\bar{g}_{x(1)}(n\theta)\right]^k\right]$$

$$= \left[\frac{u}{1-R}\right]\sum_{i=r}^{n} nC_i\left[\frac{1}{1-\bar{g}_{x(1)}(i\theta)} - (n-i)C_1\frac{1}{1-\bar{g}_{x(1)}((i+1)\theta)} + (n-i)C_2\frac{1}{1-\bar{g}_{x(1)}((i+2)\theta)} + \cdots (-1)^{n-i}\frac{1}{1-\bar{g}_{x(1)}(n\theta)}\right] \qquad (32)$$

Substituting (14) in (32)

$$\text{E(T)} = \left[\frac{u}{1-R}\right]\sum_{i=r}^{n} nC_i\left\{\left(\frac{mc+i\theta}{i\theta}\right) - (n-i)C_i\left(\frac{mc+(i+1)\theta}{(i+1)\theta}\right) + \cdots (-1)^{n-i}\left(\frac{mc+n\theta}{n\theta}\right)\right\} \qquad (33)$$

Suppose $g(x) = g_{x(m)}(x)$

Then

$$P(T > t) = \sum_{k=0}^{\infty}[F_k(t) - F_{k+1}(t)]\sum_{i=r}^{n} nC_i\left[\left[\bar{g}_{x(m)}(i\theta)\right]^k - (n-i)C_1\left[\bar{g}_{x(m)}((i+1)\theta)\right]^k + (n-i)C_2\left[\bar{g}_{x(m)}((i+2)\theta)\right]^k +\right.$$

…−1n−i[gx(m)(nθ)]k (34)

$L(t) = 1 - P(T > t)$

$l(t) = \frac{d}{dt}(L(t))$

$=$

$-\sum_{k=0}^{\infty}[f_k(t) -$

fk+1ti=rnnCigx(m)iθk−n−iC1gx(m) i+1θk+n−iC2gx(m)(i+2θ)k+ …−1n−i[gx(m)(nθ)]k (35)

$$\bar{l}(s) = -\sum_{k=0}^{\infty}[\bar{f}_k(t) - \bar{f}_{k+1}(t)] \times \sum_{i=r}^{n} nC_i\left[\left[\bar{g}_{x(m)}(i\theta)\right]^k - (n-i)C_1\left[\bar{g}_{x(m)}((i+1)\theta)\right]^k + (n-i)C_2\left[\bar{g}_{x(m)}((i+2)\theta)\right]^k +\right.$$

…−1n−i[gx(m)(nθ)]k (36)

$$\bar{l}(s) = \sum_{i=r}^{n} nC_i\left\{[1-\bar{g}_{x(m)}(i\theta)]\sum_{k=0}^{\infty}\bar{f}_k(s)\left[\bar{g}_{x(m)}(i\theta)\right]^{k-1} - (n-i)C_1[1-\bar{g}_{x(m)}((i+1)\theta)]\sum_{k=0}^{\infty}\bar{f}_k(s)\left[\bar{g}_{x(m)}((i+1)\theta)\right]^{k-1} +\right.$$

…1−gx(m)nθ𝑘=0∞𝑓𝑘sgx(m)nθk−1

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$$\bar{l}(s) = \sum_{i=r}^{n} nC_i \left\{ \frac{[1-\bar{g}_{x(m)}(i\theta)]\bar{f}(s)}{1-\bar{f}(s)\,\bar{g}_{x(m)}(i\theta)} - (n-i)C_1 \frac{[1-\bar{g}_{x(m)}((i+1)\theta)]\bar{f}(s)}{1-\bar{f}(s)\,\bar{g}_{x(m)}((i+1)\theta)} + \cdots (-1)^{n-i} \frac{[1-\bar{g}_{x(m)}(n\theta)]\bar{f}(s)}{1-\bar{f}(s)\,\bar{g}_{x(m)}(n\theta)} \right\} \quad (37)$$

Since $g(x) = ce^{-cx}$

$g_{X_{(m)}}(x) = m\, g(x)[G(x)]^{m-1}$

$g_{x(m)}(x) = mce^{-cx}(1-e^{-cx})^{m-1}$

$\bar{g}_{x(m)}(\theta) = mc\int_0^\infty e^{-cx}(1-e^{-cx})^{m-1}e^{-\theta x}dx = -m\int_1^0 (1-z)^{m-1} e^{\theta\left(\frac{\log z}{c}\right)}dz = m\int_0^1 (1-z)^{m-1} z^{\frac{\theta}{c}}dz.$

$= m\beta\left(\frac{\theta}{c}+1, m\right) = m\frac{\Gamma\left(\frac{\theta}{c}+1\right)\Gamma(m)}{\Gamma\left(\theta/c+m\right)}$

Simplifying the right side, we get

$$\bar{g}_{x(m)}(\theta) = \frac{m!c^m}{\delta(c,\theta)} \quad (38)$$

Where $\delta(c,\theta) = (c+\theta)(2c+\theta)(3c+\theta)\ldots(mc+\theta)$ (39)

$$E(T) = -\frac{d}{ds}(\bar{l}(s))_{s=0}$$

$$= -\sum_{i=r}^{n} nC_i \left\{ \frac{[1-\bar{g}_{x(m)}(i\theta)]\left[\frac{d}{ds}(\bar{f}(s)\right]_{s=0}}{[1-\bar{g}_{x(m)}(i\theta)\bar{f}(s)]_{s=0}^2} - (n-i)C_1 \frac{[1-\bar{g}_{x(m)}((i+1)\theta)]\left[\frac{d}{ds}(\bar{f}(s)\right]_{s=0}}{[1-\bar{g}_{x(m)}((i+1)\theta)\bar{f}(s)]_{s=0}^2} + \cdots \quad (-1)^{n-i} \frac{[1-\bar{g}_{x(m)}(n\theta)]\left[\frac{d}{ds}(\bar{f}(s)\right]_{s=0}}{[1-\bar{g}_{x(m)}(n\theta)\bar{f}(s)]_{s=0}^2} \right\}$$

$$= -\sum_{i=r}^{n} nC_i \left\{ \frac{1}{[1-\bar{g}_{x(m)}(i\theta)]} - (n-i)C_1 \frac{1}{[1-\bar{g}_{x(m)}((i+1)\theta)]} + \cdots (-1)^{n-i} \frac{1}{[1-\bar{g}_{x(m)}(n\theta)]} \right\} \left[\frac{d}{ds}(\bar{f}(s)\right]_{s=0} \quad (40)$$

We now obtain the analytical result for mean time to recruitment in closed form for four different cases on inter-decision times $\{U_i\}_{i=1}^{\infty}$

Case (i) $\{U_i\}_{i=1}^{\infty}$ form an Ordinary Renewal Process:

The inter decision times are assumed to be independent and identically distributed hyper exponential random variable with probability density function $f(t) = pe^{-\lambda_h t} + qe^{-\lambda_l t}, p+q=1$.Where λ_h, λ_l are high and low attrition rate p,q are the proportion of decisions having high and low attrition.

$\bar{f}(s) = \frac{p\lambda_h}{\lambda_h+s} + \frac{q\lambda_l}{\lambda_l+s}, \bar{f}(0) = \frac{p\lambda_h}{\lambda_h} + \frac{q\lambda_l}{\lambda_l} = p+q = 1, \frac{d}{ds}\left(\bar{f}(s)\right) = \frac{-p\lambda_h}{(\lambda_h+s)^2} + \frac{-q\lambda_l}{(\lambda_l+s)^2}$

$\left(\frac{d}{ds}\left(\bar{f}(s)\right)\right)_{s=0} = -\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right), \bar{f}(0) = 1$

Substituting the above equation in (40), we get

$$E(T) = \left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n} nC_i \left\{ \frac{1}{[1-\bar{g}_{x(m)}(i\theta)]} - (n-i)C_1 \frac{1}{[1-\bar{g}_{x(m)}((i+1)\theta)]} + \cdots (-1)^{n-i} \frac{1}{[1-\bar{g}_{x(m)}(n\theta)]} \right\} \quad (41)$$

Substituting (38) in (41)

$$E(T) = \left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n} nC_i \left\{ \left(\frac{\delta(c,i\theta)}{\delta(c,i\theta)-m!c^m}\right) - (n-i)C_1 \left(\frac{\delta(c,(i+1)\theta)}{\delta(c,(i+1)\theta)-m!c^m}\right) + \cdots (-1)^{n-i} \left(\frac{\delta(c,n\theta)}{\delta(c,n\theta)-m!c^m}\right) \right\} \quad (42)$$

Where $\delta(c,\theta)$is given by the equation (39)

Case (ii) $\{U_i\}_{i=1}^{\infty}$ form an order statistics

Consider the population $\{U_i\}_{i=1}^{\infty}$ of independent and identically distributed interdecision times with hyper exponential cumulative distribution $F(t) = 1 - pe^{-\lambda_h t} - qe^{-\lambda_l t}$ and the corresponding density function f(t).Assume that $\{U_i\}_{i=1}^{m_1}$ be a sample of size m_1 selected from this population .Let $U_1, U_2, \ldots, U_{m_1}$ be the order statistics corresponding to this sample with respective probability density function $f_{U_1}, f_{U_2}, \ldots\ldots f_{U_{m_1}}$. U_1 is the first order statistics and U_{m_1} is the m_1th order statistics such that $U_1 \le U_2 \le \cdots\ldots \le U_{m_1}$ and hence not independent.

Sub case (i)

Substituting (18) in (40) the mean time to recruitment is

$$E(T) = \sum_{r_1=0}^{m_1}\binom{m_1}{r_1} p^{r_1} q^{m_1-r_1}\left[\frac{1}{(\lambda_h r_1 - \lambda_l r_1 + \lambda_l m_1)}\right]\sum_{i=r}^{n} nC_i \left\{ \frac{1}{[1-\bar{g}_{x(m)}(i\theta)]} - (n-i)C_i \frac{1}{[1-\bar{g}_{x(m)}((i+1)\theta)]} + \cdots (-1)^{n-i} \frac{1}{[1-\bar{g}_{x(m)}(n\theta)]} \right\} \quad (43)$$

Substituting (38) in (43)

$$E(T) = \sum_{r_1=0}^{m_1}\binom{m_1}{r_1} p^{r_1} q^{m_1-r_1}\left[\frac{1}{(\lambda_h r_1 - \lambda_l r_1 + \lambda_l m_1)}\right] \times \sum_{i=r}^{n} nC_i \sum_{i=r}^{n} nC_i \left\{ \left(\frac{\delta(c,i\theta)}{\delta(c,i\theta)-m!c^m}\right) - (n-i)C_1 \left(\frac{\delta(c,(i+1)\theta)}{\delta(c,(i+1)\theta)-m!c^m}\right) + \right.$$

…−1n−iδ(c,nθ)δc,nθ−m!cm

(44)

Where $\delta(c,\theta)$is given by the equation (39)

Sub case (ii)

Substituting (21) in (40) the mean time to recruitment is

$$E(T) = \sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_1-r_1}\frac{m_1!}{r_1!r_2!(m_1-r_1-r_2)!}p^{r_2}q^{m_1-r_1-r_2}\frac{1}{(\lambda_h r_2 - \lambda_l r_2 - \lambda_l r_1 + \lambda_l m_1)} \times \sum_{i=r}^{n} nC_i \left\{ \frac{1}{[1-\bar{g}_{x(m)}(i\theta)]} - (n-i)C_1 \frac{1}{[1-\bar{g}_{x(m)}((i+1)\theta)]} + \right.$$

…(−1)𝑛−𝑖11−𝑔𝑥(𝑚)nθ

(45)

Substituting (38) in (45)

$E(T) = \sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_1-r_1}(-1)^{m_1-r_1+1}\frac{m_1!}{r_1!r_2!(m_1-r_1-r_2)!}p^{r_2}q^{m_1-r_1-r_2}\frac{1}{(\lambda_h r_2-\lambda_l r_2-\lambda_l r_1+\lambda_l m_1)} \times \sum_{i=r}^{n} nC_i \left\{\left(\frac{\delta(c,i\theta)}{\delta(c,i\theta)-m!c^m}\right) - (n-$

$iC1\delta(c,(i+1)\theta)\delta c,(i+1)\theta-m!cm+\ldots-1n-i\delta(c,n\theta)\delta c,n\theta-m!cm$

(46)

Where $\delta(c,\theta)$ is given by the equation (39)

Case (iii) $\{U_i\}_{i=1}^{\infty}$ form a Geometric Process:

Assume that the inter decision times $U_i, i = 1,2,3 \ldots$ form a geometric process with rate b, (b>0).It is assumed that the probability density function of U_1 is hyper exponential density function $f(t) = p\lambda_h e^{-\lambda_h} + q\lambda_l e^{-\lambda_l}, p + q = 1$.

From equation (10), we have

$\bar{l}(s) = -\sum_{k=0}^{\infty}[\bar{f}_k(s) - \bar{f}_{k+1}(s)] \times \sum_{i=r}^{n} nC_i \left[[\bar{g}_{x(m)}(i\theta)]^k - (n-i)C_1[\bar{g}_{x(m)}((i+1)\theta)]^k + (n-i)C_2[\bar{g}_{x(m)}((i+2)\theta)]^k + \right.$

$\ldots-1n-igx(m)(n\theta)k$

$$E(T) = -\left[\frac{d}{ds}(\bar{l}(s))\right]_{s=0}$$

$= \sum_{k=0}^{\infty}\left[\left[\frac{d}{ds}(\bar{f}_k(s))\right]_{s=0} - \left[\frac{d}{ds}(\bar{f}_{k+1}(s))\right]_{s=0}\right] \times \sum_{i=r}^{n} nC_i \left[[\bar{g}_{x(m)}(i\theta)]^k - (n-i)C_1[\bar{g}_{x(m)}((i+1)\theta)]^k + (n-i)C_2[\bar{g}_{x(m)}((i+\right.$

$2\theta)k+\ldots-1n-igx(m)(n\theta)k$ (47)

Substituting (26) in (47)

$=$

$\left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{k=0}^{\infty}\frac{1}{b^k}\times$

$i=rnnCigx(m)i\theta k-n-iC1gx(m)\ i+1\theta k+n-iC2gx(m)(i+2\theta)k+\ \ldots-1n-igx(m)(n\theta)k$

$$= \left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n} nC_i\sum_{k=0}^{\infty}\left[\left[\frac{\bar{g}_{x(m)}(i\theta)}{b}\right]^k - (n-i)C_1\left[\frac{\bar{g}_{x(m)}((i+1)\theta)}{b}\right]^k + (n-i)C_2\left[\frac{\bar{g}_{x(m)}((i+2)\theta)}{b}\right]^k + \ldots(-1)^{n-i}\left[\frac{\bar{g}_{x(m)}(n\theta)}{b}\right]^k\right]$$

$= \left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n} nC_i\left[\left[1-\frac{\bar{g}_{x(m)}(i\theta)}{b}\right]^{-1} - (n-i)C_1\left[1-\frac{\bar{g}_{x(m)}((i+1)\theta)}{b}\right]^{-1} + (n-i)C_2\left[1-\frac{\bar{g}_{x(m)}((i+2)\theta)}{b}\right]^{-1} + \ldots(-1)^{n-i}\left[1-\right.\right.$

$gx(m)(n\theta)b-1$

$$= \left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n} nC_i\left[\frac{b}{b-\bar{g}_{x(m)}(i\theta)}\right] - (n-i)C_1\left[\frac{b}{b-\bar{g}_{x(m)}((i+1)\theta)}\right] + (n-i)C_2\left[\frac{b}{b-\bar{g}_{x(m)}((i+2)\theta)}\right] + \cdots(-1)^{n-i}\left[\frac{b}{b-\bar{g}_{x(m)}((n\theta)}\right] \quad (48)$$

Substituting (38) in (48)

$$= \left(\frac{p\lambda_l+q\lambda_h}{\lambda_h\lambda_l}\right)\sum_{i=r}^{n} nC_i\left\{\left(\frac{b\delta(c,i\theta)}{b\delta(c,i\theta)-m!c^m}\right) - (n-i)C_i\left(\frac{b\delta(c,(i+1)\theta)}{b\delta(c,(i+1)\theta)-m!c^m}\right) + \cdots(-1)^{n-i}\left(\frac{b\delta(c,n\theta)}{b\delta(c,n\theta)-m!c^m}\right)\right\} \quad (49)$$

Where $\delta(c,\theta)$ is given by the equation (39)

Case (iv) When $U_i's$ are correlated:

The inter decision times are assumed to be exchangeable and constantly correlated exponential random variables with mean $\frac{1}{\mu}(\mu > 0)$.Let R be the constant correlation between U_i and $U_j, i \neq j$.

By taking Laplace Stieljes transform both side using (9)

$\bar{L}(s) =$

$-\sum_{k=1}^{\infty}[\bar{F}_k(s) - \bar{F}_{k+1}(s)] \times \sum_{i=r}^{n} nC_i[[\bar{g}(i\theta)]^k - (n-i)C_1[\bar{g}((i+1)\theta)]^k + (n-i)C_2[\bar{g}((i+2)\theta)]^k + \ldots(-1)^{n-i}[\bar{g}(n\theta)]^k]$

$$E(T) = -\left[\frac{d}{ds}(\bar{L}(s))\right]_{s=0}$$

$=$

$\sum_{k=0}^{\infty}\left[\left[\frac{d}{ds}(\bar{F}_k(s))\right]_{s=0} - \left[\frac{d}{ds}(\bar{F}_{k+1}(s))\right]_{s=0}\right] \times \sum_{i=r}^{n} nC_i\left[[\bar{g}_{x(m)}(i\theta)]^k - (n-i)C_1[\bar{g}_{x(m)}((i+1)\theta)]^k + (n-i)C_2[\bar{g}_{x(m)}((i+\right.$

$2\theta k\ \ldots-1n-igx(m)n\theta k$ (50)

$E(T) =$

$\sum_{k=0}^{\infty}\left[\frac{u}{1-R}\right]\sum_{i=r}^{n} nC_i\left[[\bar{g}_{x(m)}(i\theta)]^k - (n-i)C_1[\bar{g}_{x(m)}((i+1)\theta)]^k + (n-i)C_2[\bar{g}_{x(m)}((i+2)\theta)]^k\ \ldots(-1)^{n-i}[\bar{g}_{x(m)}(n\theta)]^k\right]$

Substituting (31) in (50)

$E(T) =$

$\left[\frac{u}{1-R}\right]\sum_{i=r}^{n} nC_i\sum_{k=0}^{\infty}\left[[\bar{g}_{x(m)}(i\theta)]^k - (n-i)C_1[\bar{g}_{x(m)}((i+1)\theta)]^k + (n-i)C_2[\bar{g}_{x(m)}((i+2)\theta)]^k\ \ldots(-1)^{n-i}[\bar{g}_{x(m)}(n\theta)]^k\right]$

$$= \left[\frac{u}{1-R}\right]\sum_{i=r}^{n} nC_i\left[\frac{1}{1-\bar{g}_{x(m)}(i\theta)} - (n-i)C_1\frac{1}{1-\bar{g}_{x(m)}((i+1)\theta)} + (n-i)C_2\frac{1}{1-\bar{g}_{x(m)}((i+2)\theta)} + \cdots(-1)^{n-i}\frac{1}{1-\bar{g}_{x(m)}(n\theta)}\right] \quad (51)$$

Substituting (31) in (51)

$$E(T) = \left[\frac{u}{1-R}\right]\sum_{i=r}^{n} nC_i\left\{\left(\frac{\delta(c,i\theta)}{\delta(c,i\theta)-m!c^m}\right) - (n-i)C_i\left(\frac{\delta(c,(i+1)\theta)}{\delta(c,(i+1)\theta)-m!c^m}\right) + \cdots(-1)^{n-i}\left(\frac{\delta(c,n\theta)}{\delta(c,n\theta)-m!c^m}\right)\right\} \quad (52)$$

Conclusion:

From the present work we can study about two grade and three grade manpower system. This work also can be extended in two sources of depletion. The influence of the hypothetical parameter on the performance measure can be studied numerically with the help of MATLAB by fixing the value for n and r.

References:

1. Barthlomew. D. J, and Forbes. A. F, Statistical techniques for manpower planning, John Wiley & Sons,(1979).
2. Grinold.R.C, and Marshall. K. J, Man Power Planning, North Holland, Newyork (1977).
3. Sridharan.J, Parameswari. K and Srinivasan. A, A stochastic model on time to recruitment in a two grade manpower system based on order statistics, International Journal of Mathematical Sciences and Engineering Applications 6(5)(2012):23-30.
4. Sridharan. J, Parameswari. K and Srinivasan. A, A stochastic model on time to recruitment in a two grade manpower system involving extended exponential threshold based on order statistics, Bessel Journal ofMathematics3(1) (2013):39-49.
5. Sridharan. J, Parameswari. K and Srinivasan. A, A stochastic model on time to recruitment in a two grade manpower system involving extended exponential and exponential threshold based on order statistics, Archimedes Journal of Mathematics3(1) (2013):41-50
6. Sridharan. J, Parameswari. K and Srinivasan. A, A stochastic model on time to recruitment in a two grade manpower system basedon order statistics when the threshold distribution having SCBZ property, Cayley Journal of Mathematics 1(2) (2012): 101-112
7. Parameswari. K, Sridharan. J and Srinivasan. A, Time to recruitment in a two grade manpower system based on order statistics, Antarctica Journal of Mathematics 10(2) (2013):169-181.
8. Parameswari. K and Srinivasan. A, Estimation of variance of time to recruitment for a two grade manpower system with two types of decisions when the wastages form a geometric process, International Journal of mathematics Trends and Technology (IJMTT) – Volume 33 Number 3- May 2016
9. S. Dhivya, V. Vasudevan and A. Srinivasn, Stochastic models for the time to recruitment in a two grade manpower system using same geometric process for the inter decision times, proceedings of mathematical and computational models, PSG college of technology(ICMCM),Narosa publishing House,pp.276-283,Dec -2011
10. Vidhya. S, A study on some stochastic models for amulti graded manpower system, Ph.D thesis, Bharathidasan University (2011).
11. K. Srividhya, S. Sendhamizhi Selvi, Mean Time to Recruitment for a Multi Grade Manpower system with single threshold, single source of depletion when interpolicy decisions form an order statistics, IOSR Journal of Mathematics vol 13,issue 3 pp33-38
12. K. Srividhya, S. Sendhamizhselvi "Mean Time to Recruitment for a Multi Grade Man Power System with Single Threshold, Single Source of Depletion when Inter Policy Decisions form a Geometric Process", International Journal of Mathematics Trends and Technology (IJMTT). *V45* (1):11-15 May 2017. ISSN: 2231-5373.