ISSN: 2455 – 5428
International Journal of Current Research and Modern Education
Impact Factor 6.725, Special Issue, July - 2017

STABILIZATION OF DISCRETE STOCHASTIC DYNAMIC SYSTEM WITH DELAY

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Cite This Article: S. Elizabeth & S. Nirmal Veena, "Stabilization of Discrete Stochastic Dynamic System with Delay", International Journal of Current Research and Modern Education, Special Issue, July, Page Number 53-56, 2017.

Abstract:

In this paper, we discuss the stability of stochastic type differential equations through obtaining the stability condition for the respective stochastic difference equation. The system formulation is done by considering the stochastic differential equation that describes the dynamics of single isolated neuron involving delay. Here the discretization of the stochastic differential equation is done through the Euler- Maruyama Method. And the desired stability is obtained by applying suitable assumptions and through the help of theorems. The obtained theoretical results are represented through numerical simulation using MATLAB

Key Words: Difference Equation, Stochastic Difference Equation, Martingale Sequence, Lyapunov-Krasovkii Functional & Neural Networks.

1. Introduction:

Difference equations have attracted much attention due to theirapplications in many areas of real world problems. Inparticular, difference equation in neural networks with delays have established many applications in specific fields such as signal processing, image processing, pattern recognition, associative memoryand optimization problems. These applications strongly depend on the stability of the equilibrium point of the system designed. Thus, the stability analysis is necessaryfor the design and applications in digital signal processing. Inhardware enactment of recurrent neural networks, timedelays happen due to finite switching speed of the amplifiersand communication time. In current years, significant number ofresearch works has been rendered to study the global asymptotic orexponential stability for the neural networks with time delaysvia Lyapunov function method. Especially, growinginterest exists in the study of differential equations with both discrete and distributed delays, see [5, 6]. And also, during the execution of the computation, there aremany stochastic disturbances that disturb the stability of a designed system.A designed system could be stabilized or destabilizedby convinced stochastic inputs. However, besides stochastic effects, impulsiveeffects as well exist in real systems. Therefore, it is essential to ruminate both impulsive and stochastic effect on system of difference equations [10, 11]. Inthe case of linear stochastic differential equations the mean-square asymptotic stabilityof the numerical method has been studied by several authors e.g., [6, 11]. But the almost sure asymptotic stability ofnumerical method has been less studied. Inspired by the above discussions, the mainobjective of this paper is to study the global asymptoticstability of system of difference equation describing the dynamics of a neuron.In this paper we consider the almost sure asymptotic stability of the strong Euler–Maruyama method to the nonlinear scalar stochastic differential equation.It shows that achieved difference equation is a worthy discrete model, since under the conditions from Theorem 4.1, solutions of the continuous problem have the same asymptoticbehavior. We establish new stabilityconditions for the stochastic difference equation with the help of Lyapunov-Krasovskii functional method and somewell-known inequalities. An example with numerical simulation results are givento show the effectiveness of the proposed stability result.

2. Preliminaries:

In this section we recall the necessary definitions and lemmas that will be used to prove our results. Take $(\Omega,\mathcal{F},\{\mathcal{F}_k\}_{k\in\mathbb{N}},\mathbb{P})$tobe a complete filtered probability space. Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence of independent random variables with $\boldsymbol{E}\xi_k = 0$ and $\boldsymbol{E}{\xi_k}^2 = 1$. We assume that our filter $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$ is is naturally generated and takes the form $\mathcal{F}_{k+1} = \sigma\{\xi_{i+1}, i - 0,1,2 \dots n\}$.

Here we consider a sequence of random variables $\{Y_k\}_{k\in\mathbb{N}}$. A discrete time stochastic sequence $\{Y_k\}_{k\in\mathbb{N}}$ is said to $\mathcal{F}_k - martingale$ if it statisfies for any time $k, \boldsymbol{E}(Y_k|\mathcal{F}_{k-1}) < \infty$ and $\boldsymbol{E}(Y_{k+1}|\mathcal{F}_k) = Y_{k+1}$

A real Valued process Y defined on a filtered space $(\Omega,\mathcal{F},\{\mathcal{F}_k\}_{k\in\mathbb{N}},\mathbb{P})$ is called a $seimartingale$ if it can be decomposed as $Y_k = M_k + A_k$ where M is a local martingale A is adapted process of a bounded variation.

And the stochastic sequence $\{\xi_k\}_{k\in\mathbb{N}}$ is said to be $\mathcal{F}_k - martingale\ difference$, if $E(|\xi_k|) < \infty$ an $E(\xi_{k+1}|\mathcal{F}_k) = 0$ almost surely $\forall k \in \mathbb{N}$.

3. Mathematical Formulation:

The following represents the system of stochastic differential equation which describes the dynamics of an isolated neuron described in terms of delay differential equations

$$dy(t) = [-y(t) + \alpha\varphi(y(t) - \beta y(t-\tau)]dt + \psi\big(t, y(t-\tau)\big)dW_t, \qquad t \geq 0 \qquad (1)$$

Where $y(t) -$ is the activation level of a neuron at a time t

α −is constant describing the range of the variable $y(t)$

β −is the measure that describes the influence of past history

τ −represents the delay

φ −activation function of the neuron

ψ −noise intensity.

W_t −Wiener process

and the constants $\alpha \in \mathbb{R}^+, \beta \geq 0$ and $\tau \in [0,\infty)$. And the above system forms a model of neural network.

The stability of the above system is obtained by considered the related discretized form of equation. Hence the discretization of the above model through Euler-Maruyama is given by

$$Y_{k+1} = (1-d)Y_k + \alpha d\phi(Y_k - \beta Y_{k-\tau}) + \sqrt{d}\psi_{k,Y_{k-\tau}}\xi_{k+1} \qquad , k \in \mathbb{N}_0 \qquad (2)$$

With$Y_0 \in \mathbb{R}$ -as arbitrary nonrandom initial value. $d \in (0,1]$is the mesh size. ϕ −is continous real valued function such that

$$|\varphi(x)| \le |x| \qquad (3)$$

$$\text{And } |\alpha|(1+|\beta|) < 1 \qquad (4)$$

ξ_k −are independent random variables suitably chosen with mean $E\xi_k = 0$ and variance given by unit, $E{\xi_k}^2 = 1$.

4. Main Results:

Lemma 4.1: Let $\{x_k\}_{k\in\mathbb{N}}$be a sequence of independent $\mathcal{F}_k$-measurable random variables,$E[x_k] = 0$and $\boldsymbol{E}|x_k| < \infty = 0$,. Let also $\{y_k\}_{k\in\mathbb{N}}$ be a sequence of $\mathcal{F}_k$-measurable randomvariables such that$\boldsymbol{E}|y_{k-1}|x_k| < \infty$for all $k \ge 1$. Then$\{Z_k\}_{k\in\mathbb{N}}$,

$$Z_k = \sum_{s=1}^{k} y_{s-1}x_s$$

For all $k \in \mathbb{N}$, is an $\mathcal{F}_k$- *martingale* and $\{y_{k-1}x_k\}$ is an $\mathcal{F}_k$- *martingale difference*.

Lemma 4.2: Let $\{W_k\}_{k\in\mathbb{N}}$ be a non-negative $\mathcal{F}_k - martingale$ process, $\mathbb{E}|W_k| < \infty\ \forall k \in \mathbb{N}$ and

$$W_{k+1} \le W_k + u_k - v_k + v_{k+1}\ , \qquad k \in \mathbb{N}_{0,}$$

Where $\{v_k\}_{k\in\mathbb{N}}$ is an$\mathcal{F}_k - martingale\ difference$, $\{u_k\}_{k\in\mathbb{N}}, \{v_k\}_{k\in\mathbb{N}}$are nonnegative$\mathcal{F}_k - martingale$ process, $\mathbb{E}|u_k|, \mathbb{E}|v_k| < \infty\ \forall k \in \mathbb{N}$. Then

$$\{\omega: \textstyle\sum_{k=1}^{\infty} u_k < \infty\} \subseteq \{\omega: \sum_{k=1}^{\infty} v_k < \infty\} \cap \{W_k \to\}.$$

By $\{W_k \to\}$ we refer to the set of all $\omega \in \Omega$, such that$\lim_{k\to\infty} W_k(\omega) = 0$ exists and is finite.

Theorem 4.1: Let $Y_k$be the solution of the equation (2) with the conditions

$$|\psi_{k,Y_k}| \le \gamma_k|Y_k|^2 + {\eta_k}^2, \qquad \psi_{k,0} = 0 \qquad (5)$$

Where $\sum_{i=1}^{\infty} {\eta_i}^2 < \infty$ and

$$\alpha^2(1+|\beta|)^2 + \gamma_k < 1 \qquad (6)$$

Satisfied by equation (4).Then$\lim_{k\to\infty} Y_k = 0$ almost everywhere.

Proof: Consider equation (2),

$$Y_{k+1} = (1-d)Y_k + \alpha d\phi(Y_k - \beta Y_{k-\tau}) + \sqrt{d}\psi_{k,Y_{k-\tau}}\xi_{k+1} \qquad , k \in \mathbb{N}_0$$

Squaring on both sides we get

$$\begin{aligned} {Y_{k+1}}^2 &= [(1-d)Y_k + \alpha d\phi(Y_k - \beta Y_{k-\tau})]^2 + 2[(1-d)Y_k + \alpha d\phi(Y_k - \beta Y_{k-\tau})]\sqrt{d}\psi_{k,Y_{k-\tau}}\xi_{k+1} + d\psi^2{}_{k,Y_{k-\tau}}{\xi_{k+1}}^2 \\ &= [(1-d)Y_k + \alpha d\phi(Y_k - \beta Y_{k-\tau})]^2 + d\psi^2{}_{k,Y_{k-\tau}} + \delta_{k+1}. \qquad (7) \end{aligned}$$

Where $\{\delta_k\}_{k\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale difference, and its value is defined as

$$\delta_{k+1} = 2[(1-d)Y_k + \alpha d\phi(Y_k - \beta Y_{k-\tau})]\sqrt{d}\psi_{k,Y_{k-\tau}}\xi_{k+1} + d\psi^2{}_{k,Y_{k-\tau}}\{{\xi_{k+1}}^2 - 1\} \qquad (8)$$

Also from our assumption we have $E{\xi_k}^2 = 1$. Hence $\{E{\xi_k}^2 - 1\}$ $\mathcal{F}_{k+1}$-martingale-difference. And from the lemma we conclude that δ_{k+1} is $\mathcal{F}_{k+1}$-martingale-difference.

Consider

$$\sum_{i=1}^{n} x_iy_i \le \sum_{i=1}^{n} |x_iy_i| \le \sum_{i=1}^{n} \sqrt{|x_i|}(\sqrt{|x_i|}\,|y_i|)$$

Now from Holder's inequality we have

$$\sum_{i=1}^{n} |x_iy_i| \le \sum_{i=1}^{n} \sqrt{|x_i|}(\sqrt{|x_i|}\,|y_i|) \le \sqrt{\sum_{i=1}^{n}|x_i|}\sqrt{\sum_{i=1}^{n}|x_i|\,|y_i|^2}$$

Now consider,

$$\begin{aligned} |(1-d)Y_k + \alpha d\phi(Y_k - \beta Y_{k-\tau})| &\le |(1-d)Y_k| + |\alpha|d|\phi(Y_k - \beta Y_{k-\tau})| \\ &\le |(1-d)Y_k| + |\alpha|d|Y_k - \beta Y_{k-\tau}| \\ &\le (1-d)|Y_k| + |\alpha|d[|Y_k| + |\beta||Y_{k-\tau}|] \\ &\le (1-d+|\alpha|d)|Y_k| + |\alpha|d|\beta||Y_{k-\tau}| \end{aligned}$$

Here let $n = 2 x_1 = 1-d+|\alpha|d, \quad y_1 = |Y_k|, \quad x_2 = |\alpha|d|\beta|, \quad y_2 = |Y_{k-\tau}|,$

$$\le [1-d+|\alpha|d+|\alpha|d|\beta|][(1-d+|\alpha|d)|Y_k|^2 + |\alpha|d|\beta||Y_{k-\tau}|^2] \qquad (9)$$

Substituting (9) in (7) we have

$$\begin{aligned} {Y_{k+1}}^2 &\le [1-d+|\alpha|d+|\alpha|d|\beta|][(1-d+|\alpha|d)|Y_k|^2 + |\alpha|d|\beta||Y_{k-\tau}|^2] + d\psi^2{}_{k,Y_{k-\tau}} + \delta_{k+1} \\ &\le [1-d+|\alpha|d+|\alpha|d|\beta|][(1-d+|\alpha|d)|Y_k|^2 + |\alpha|d|\beta||Y_{k-\tau}|^2] + d\gamma_k|Y_k|^2 + d{\eta_k}^2 + \delta_{k+1} \\ &\le [(1-d+|\alpha|d+|\alpha|d|\beta|)][(1-d+|\alpha|d)|Y_k|^2 + |\alpha|d|\beta||Y_{k-\tau}|^2] + d\gamma_k|Y_k|^2 + d{\eta_k}^2 + \delta_{k+1} \\ &\le [(1-d+|\alpha|d+|\alpha|d|\beta|)(1-d+|\alpha|d) + d\gamma_k]|Y_k|^2 + [d|\alpha||\beta|(1-d+|\alpha|d+|\alpha|d|\beta|)]|Y_{k-\tau}|^2 + d{\eta_k}^2 + \delta_{k+1} \end{aligned}$$

Let $a = d|\alpha||\beta|(1-d+|\alpha|d+|\alpha|d|\beta|)$

Let

$$V(k)^{(1)} = a\sum_{s=k-\tau}^{k-1} {Y_s}^2$$

$$V(k) = {Y_k}^2 + V(k)^{(1)}$$

Consider, $\Delta V(k)^{(1)} = V(k+1)^{(1)} - V(k)^{(1)}$

$$\begin{aligned} &= a\sum_{s=k+1-\tau}^{k} {Y_s}^2 - a\sum_{s=k-\tau}^{k-1} {Y_s}^2 \\ &= a{Y_k}^2 - a{Y_{k-\tau}}^2 \end{aligned}$$

Now consider

$$\Delta V(k) = {Y_{k+1}}^2 - {Y_k}^2 + \Delta V(k)^{(1)}$$
$$= {Y_{k+1}}^2 - {Y_k}^2 + a{Y_k}^2 - a{Y_{k-\tau}}^2$$
$$\leq [(1-d+|\alpha|d+|\alpha|d|\beta|)(1-d+|\alpha|d)+d\gamma_k]|Y_k|^2 + [d|\alpha||\beta|(1-d+|\alpha|d+|\alpha|d|\beta|)]|Y_{k-\tau}|^2 + d{\eta_k}^2 + \delta_{k+1} - {Y_k}^2 + a{Y_k}^2 - a{Y_{k-\tau}}^2$$
$$\leq [a-1+(1-d+|\alpha|d+|\alpha|d|\beta|)(1-d+|\alpha|d)+d\gamma_k]|Y_k|^2 + d{\eta_k}^2 + \delta_{k+1}$$
$$\leq [d|\alpha||\beta|(1-d+|\alpha|d+|\alpha|d|\beta|)-1+(1-d+|\alpha|d+|\alpha|d|\beta|)(1-d+|\alpha|d)+d\gamma_k]|Y_k|^2 + d{\eta_k}^2 + \delta_{k+1}$$
$$\leq \left(\left[1-d\left(1-|\alpha|(1+|\beta|)\right)\right]^2 - 1 + d\gamma_k\right)|Y_k|^2 + d{\eta_k}^2 + \delta_{k+1}) \qquad (10)$$

From (6) and for all $d \in (0,1]$ we get,
$0 < 1-|\alpha|(1+|\beta|) < 1, 0 < d[1-|\alpha|(1+|\beta|)] < 1$
$$\Rightarrow 0 < 1-d[1-|\alpha|(1+|\beta|)] < |\alpha|(1+|\beta|)$$
Therefore, $(1-d[1-|\alpha|(1+|\beta|)])^2 + d\gamma_k \leq \alpha^2(1+|\beta|)^2 + d\gamma_k < 1$.
Let us denote $\xi = \left(1-\left[1-d\left(1-|\alpha|(1+|\beta|)\right)\right]^2 - d\gamma_k\right)$
From (10), we have $\Delta V(k) \leq \left(\left[1-d\left(1-|\alpha|(1+|\beta|)\right)\right]^2 - 1 + d\gamma_k\right)|Y_k|^2 + d{\eta_k}^2 + \delta_{k+1}$
$$V(k+1) \leq V(k) - \xi{Y_k}^2 + d{\eta_k}^2 + \delta_{k+1} \qquad (11)$$
And let, $W_k = V(k)$, $u_k = d{\eta_k}^2$, $v_k = \xi{Y_k}^2$ and $v_{k+1} = \delta_{k+1}$
Applying lemma 4.2 we have,
$$\lim_{k\to\infty} V(k) \text{ and } \xi \lim_{k\to\infty} \sum_{s=1}^{k} {Y_s}^2$$
Exists and are almost surely finite
Our intension is to prove $\lim_{k\to\infty} Y_k = 0$. Let us assume that suppose $\lim_{k\to\infty} Y_k \neq 0$ with non zero probability. Then we can find a set Ω such that ${Y_{k_l}}^2 > \varsigma(x), l \in \mathbb{N}, x \in \Omega$
Define,
$$\psi(k,x) = number\ of\ sequence\ \{k_n(x)\} \leq k$$
For all $k \in \mathbb{N}, x \in \Omega$. And $\psi(k,x) \to \infty$as $k \to \infty$. Hence we arriveat a contradiction, for $x \in \Omega$
$$\infty > \xi \sum_{s=1}^{k} {Y_s}^2(x) \geq \xi \sum_{s_l \leq k} {Y_{s_l}}^2(x) \geq \varsigma(x)\xi\psi(k,x) \to \infty, \ k \to \infty$$
This is a contradiction. Therefore we have
$$\lim_{k\to\infty} Y_k = 0$$
Hence the theorem is proved

5. Numerical Example:

In this section, we provide a numerical example todemonstrate the effectiveness of the proposed asymptoticstability result.

Example 5.1 Consider the following two-neuron stochasticrecurrent neural networks with impulses:
$$dy(t) = [-y(t) + \alpha\varphi(y(t) - \beta y(t-\tau)]dt + \psi(t, y(t-\tau))dW_t, \qquad t \geq 0, t \neq t_k \quad (12)$$
$\Delta y(t_k) = dy(t_k), t = t_k, k = 1,2,\ldots$

The activation function of the neuron is given by $\varphi(x) = \tanh(0.7) - 0.1\, sin\, x$, delay $\tau(t) = 05 + 0.5 \sin t$, $\alpha = \begin{bmatrix} -0.4 & 0.3 \\ 0.5 & 0.1 \end{bmatrix}$, $\beta = \begin{bmatrix} 0.2 & -0.4 \\ -0.1 & 0.5 \end{bmatrix}$

And here it is clear that the activation function and constants satisfies our assumptions.

Numerical Simulation:

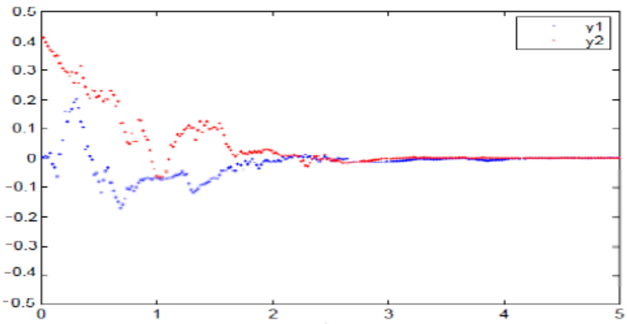

Figure 2: State responses of $y_1(t), y_2(t)$of the system (12) Withimpulsive effects.

Thus the system (12) satisfies all the conditions stated in Theorem 4.1. Hence the given stochastic neural network (12) with the impulsive effect is globally asymptotically stable.

6. Conclusion:

In this paper, the problem of stability behaviorfor a system of stochastic difference equation that describes the dynamics of single isolated neuron involving delay has been investigated by the use of Lyapunov method. By constructing an appropriate Lyapunov function and combined with stochasticanalysis approach, a new set of sufficient conditions havebeen obtained to confirm the global asymptotic stability of the addressed neural networks.Further this paper can alsobe extended to study the global exponential stability of theequilibrium point. Finally, a numerical example is given toshow the effectiveness of our stability result.

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