ISSN: 2455 – 5428
International Journal of Current Research and Modern Education
Impact Factor 6.725, Special Issue, July - 2017

ALGEBRAIC STRUCTURE OF UNION OF FUZZY SUBGROUPS AND FUZZY SUB-BIGROUPS

N. Duraimanickam* & N. Deepica**
* Assistant Professor of Mathematics, S.T.E.T Women's College, Mannargudi, Tamilnadu
** II M.Sc., Department of Mathematics, S.T.E.T Women's College, Mannargudi, Tamilnadu

Cite This Article: N. Duraimanickam & N. Deepica, "Algebraic Structure of Union of Fuzzy Subgroups and Fuzzy Sub-Bigroups", International Journal of Current Research and Modern Education, Special Issue, July, Page Number 95-99, 2017.

Abstract:

In this Paper, we discuss the definitions of union of Fuzzy subsets, Fuzzy subgroup, properties, union of Fuzzy subgroups and related theorems and proofs.

Key Words: Fuzzy Group**,** Fuzzy Sub-Bigroup & Fuzzy Subgroup

1. Introduction:

Fuzzy Subgroups:

Rosenfield introduced the notion of fuzzy group and showed that many group theory results can be extended in an elementary manner to develop the theory of fuzzy group .The underlying logic of the theory of fuzzy group is to provide a strict fuzzy algebraic structure where level subset of a fuzzy group of a group G is a subgroup of the group.

Fuzzy Sub-Bigroups:

In this section we define Fuzzy sub-bigroup of a bigroup. To define the notion of fuzzy sub-bigroup of a bigroup. We define a new notion called the fuzzy union of any two fuzzy subsets of two distinct sets.

Concept of a Fuzzy Set:

The concept of a fuzzy set is an extension of the concept of a crisp set. Just as a crisp set on a universal set U is defined by its characteristic function from U to {0,1}, a fuzzy set on a domain U is defined by its membership function from U to [0,1] .

Let U be a non-empty set, to be called the Universal set (or) Universe of discourse or simply a domain. Then, by a fuzzy set on U is meant a function A: U→[0,1]. A is called the membership function; A(x) is called the membership grade of x in A. We also write A={(x,A(x)): x∈U}.

Examples:

Consider U={a,b,c,d} and A:U→1 defined by A(a) = 0,A(b) = 0.7,A(c) = 0.4, and A(d)= 1. Then A is a fuzzy set can also be written as follows:

A={(a,0),(b,0.7),(c,0.4),(d,1)}.

Relation Between Fuzzy Sets:

Let U be a domain and A, B be fuzzy sets on U. Inclusion (or) Containment: A is said to be included (or) contained in B if and only if A(x)≤B(x) for all x in U. In symbols, we write, A ⊆ B. We also say that A is a subset of B.

Definition of Union of Fuzzysets: 1

The union of two fuzzy subsets μ_1, μ_2 is defined by $(\mu_1 \cup \mu_2)(x) = \max\{\mu_1(x),\mu_2(x)\}$ for every x in U.

Definition of Fuzzy Subgroup: 2

Let G be a group. A fuzzy subset μ of a group G is called a fuzzy subgroup of the group G if

i) $\mu(xy) \geq \min\{\mu(x),\mu(y)\}$ for every x,y∈G and

ii) $\mu(x^{-1}) = \mu(x)$ for every x∈G.

Definition of Improper Fuzzy Subgroup: 3

A fuzzy subgroup μ of a group G is called improper Fuzzy Subgroup if μ is a Constant on the group G, otherwise μ is termed as proper .

2. Fuzzy Sub-Bigroup of a Group:

Definition of Fuzzy Union of the fuzzy sets $\mu_1$andμ_2: 4

Let $\mu_1$be a fuzzy subset of a set $x_1$and μ_2 be a fuzzy subset of a set x_2 , then the fuzzy union of the fuzzy sets $\mu_1$and μ_2 is defined as a function.

$\mu_1 \cup \mu_2 : x_1 \cup x_2 \rightarrow [0,1]$ given by

$$(\mu_1 \cup \mu_2)(x) = \begin{cases} \max(\mu_1(x),\mu_2(x)) & \text{if } x \in x_1 \cap x_2. \\ \mu_1(x) & \text{if } x \in x_1 \,\&\, x \notin x_2. \\ \mu_2(x) & \text{if } x \in x_2 \,\&\, x \notin x_1. \end{cases}$$

We illustrate this definition by the following example:

Example:

Let X_1 ={ 1,2,3,4,5 } and X_2 ={2,4,6,8,10 } be two sets.

Define $\mu_1 : X_1 \rightarrow [0, 1]$ by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = 1,2. \\ 0.6 & \text{if } x=3. \\ 0.2 & \text{if } x=4,5. \end{cases}$$

and define $\mu_2 : X_2 \to [0,1]$ by

$$\mu_2(x) = \begin{cases} 1 & \text{if } x=2, 4. \\ 0.6 & \text{if } x=6. \\ 0.2 & \text{if } x = 8, 10. \end{cases}$$

It is easy to calculate $\mu_1 \cup \mu_2$ and it is given as follows:

$$(\mu_1 \cup \mu_2)(x) = \begin{cases} 1 & \text{if } x = 1,2,4. \\ 0.6 & \text{if } x = 3,6. \\ 0.2 & \text{if } x = 5, 8, 10. \end{cases}$$

Definition of Fuzzy sub- bigroup: 5

Let $G = (G_1 \cup G_2, +, \cdot)$ be a bigroup. Then $\mu: G \to [0, 1]$ is said to be a Fuzzy sub- bigroup of the bigroup G if there exists two fuzzy subsets μ_1 (of G_1) and μ_2 (of G_2) such that

i) $(\mu_1, +)$ is a fuzzy subgroup of $(G_1, +)$
ii) $(\mu_2, \cdot)$ is a fuzzy subgroup of $(G_2, \cdot)$ and
iii) $\mu = \mu_1 \cup \mu_2$.

We illustrate this by the example:

Example:

Consider the bigroup $G = \{\pm i, \pm 0, \pm 1, \pm 2, \pm 3, \ldots\}$ under the binary operation '+' and '·' where $G_1 = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ and $G_2 = \{\pm i, \pm 1\}$. Define $\mu : G \to [0, 1]$ by

$$\mu(x) = \begin{cases} \frac{1}{3} & \text{if } x = i, -i. \\ 1 & \text{if } x \in \{0, \pm 2, \pm 4, \ldots\} \\ \frac{1}{2} & \text{if } x \in \{\pm 1, \pm 3, \ldots\} \end{cases}$$

It is easy to verify that μ is a fuzzysub- bigroup of the bigroup G, for we can find $\mu_1 : G_1 \to [0, 1]$ by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x \in \{0, \pm 2, \pm 4, \ldots\} \\ \frac{1}{2} & \text{if } x \in \{\pm 1, \pm 3, \ldots\} \end{cases}$$

and $\mu_2 : G_2 \to [0, 1]$ is given by

$$\mu_2(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1, -1 \\ \frac{1}{3} & \text{if } x = i, -i \end{cases}$$

i.e ., there exists two fuzzy subgroup μ_1 of G_1 and μ_2 of G_2 such that $\mu = \mu_1 \cup \mu_2$.

3. Theorems:

Theorem 1:

μ is a fuzzy subgroup of a group S if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for all x, y in S.

Proof:

Necessary part:

Let μ is a fuzzy subgroup,

To prove: $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$.

$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y^{-1})\}$$

$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$.

Sufficient part:

$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$.

To prove: μ is a fuzzy subgroup,

$\mu(y^{-1}) = \mu(ey^{-1}) \geq \min\{\mu(e), \mu(y)\} = \mu(y)$.

$\mu(xy) = \mu(x(y^{-1})-^{1}) \geq \min\{\mu(x), \mu(y^{-1})\}$.

$\mu(xy)) \geq \min\{\mu(x), \mu(y)\}$.

Therefore μ is a fuzzy subgroup,

Hence proved.

Theorem 2:

$G_\mu = \{x \in G \,/\, \mu(x) = \mu(e)\}$ is a subgroup .

Proof:

$H = G_\mu$ and $x, y \in H$.

Then

$\mu(x) = \mu(e)$ and $\mu(y) = \mu(e)$.

$\mu(xy) \geq \min\{\mu(x), \mu(y)\} = \mu(e)$.

$\mu(xy) \geq \mu(e)$ ……………. (1)

But $\mu(e) \geq \mu(xy)$. …………...(2)

From (1) and (2),

$\mu(xy) = \mu(e)$.

and $x\,y \in H$.Also , since

$\mu(x^{-1}) = \mu(x) = \mu(e)$ implies $x^{-1} \in H$.

So, H is a subgroup.

Theorem 3:

$\mu(xy^{-1}) = \mu(e)$ implies $\mu(x)= \mu(y)$, Where μ is a subgroup .

Proof:

$\mu(x) = \mu(xy^{-1}y) \geq \min(\mu(xy^{-1}), \mu(y))$.

$\geq \min(\mu(e), \mu(y)) = \mu(y)$.

$\mu(x) \geq \mu(y)$…………(3)

$\mu(y) = \mu(yx^{-1}x) \geq \min(\mu(yx^{-1}), \mu(x))$

$= \min(\mu(xy^{-1})^{-1}, \mu(x))$.

Since μ is a subgroup.

$(\mu(xy^{-1})^{-1}) = \mu(xy^{-1})$,

so, $\mu(y) \geq \min(\mu(xy^{-1}), \mu(x))$.

$= \min(\mu(e), \mu(x))$.

$\mu(y) \geq \mu(x)$……….(4)

From (3) and (4),

$\mu(x) = \mu(y)$. Hence proved

Theorem 4:

The intersection of any collection of fuzzy subgroups is a fuzzy subgroup.

Proof:

If $\{\mu_i\}$ is a set of fuzzy subgroups . then,

$\bigcap_i \mu_i(xy) \geq \min(\bigcap_i \mu_i(x), \bigcap_i \mu_i(y))$.

$(\bigcap_i \mu_i)\, x^{-1} = \inf_i \mu_i(x^{-1}) \geq \inf_i \mu_i(x) = (\bigcap_i \mu_i)(x)$.

Theorem 5:

Let μ be a fuzzy subgroup of G. Then $\mu(x^{-1}) = \mu(x)$ and $\mu(x) = \mu(e)$ for all $x \in G$ where e is the identity element of G .

Proof:

The first part follows from

$$\mu(x) = \left[\mu\,(x^{-1})^{-1} \left[\geq \right] \mu(x^{-1}) \geq \mu(x) \right]$$

Hence, for $x \in G$.

$\mu(e) = \mu(xx^{-1})$

$\geq \min(\mu(x), \mu(x^{-1})) \geq \mu(x)$.

Hence proved.

Theorem 6:

The Characteristic function 1_T is a fuzzy subgroup if and only if T is a subgroup of G.

Proof:

If 1_T is fuzzy subgroup ,

$1_T(x\,y) \geq \min(1_T(x), 1_T(y))$…………………(5)

and $1_T(x^{-1}) \geq 1_T(x)$.

So, for $x, y \in T$, $1_T(x\,y) = 1$ and $1_T(x^{-1}) = 1$……………(6)

Hence , $x\,y \in T$ and $x^{-1} \in T$.

This proves that T is a subgroup of G.

Conversely,

if for $x, y \in G$, Conditions (1) and (2) holds if $x \notin T$ or $y \notin T$.

In the case $x \in T$, $y \in T$ for the subgroup T , We have $x^{-1} \in T$ and $x\,y \in T$.

So,(5), and (6) are equalities in the case.

Corollary:

The fuzzy subgroup generated by Characteristic function of a set is just the Characteristic function of the subgroup generated by the set , that is $(1_A) = 1_{(A)}$

Theorem 7:

A group cannot be the union of two proper fuzzy subgroup

Proof:

Let $S = \mu_1 \cup \mu_2$ and μ_1 and μ_2 be two proper fuzzy subgroups of S.

From $1 = S(x) = \mu_1(x) \vee \mu_2(x)$ either $\mu_1(x) = 1$ or $\mu_2(x) = 1$.

Let u ,v be in S such that $\mu_1(u) = 1$, μ_2 (v)< 1, $\mu_1(v) = 1$ and μ_2 (v) = 1 and

Consider u v , If $\mu_1(u\ v) = 1$, then since $\mu_1(\mu^{-1}) = 1$. We have

$\mu_1(v) = \mu_1(\mu^{-1}(u\ v) \geq \min(\mu_1(\mu^{-1}), \mu_1(u\ v)) = 1$

Hence , $\mu_1(v) = 1$, a contradiction.

A Similar contradiction is obtained if μ_2 (u v) = 1.

Theorem 8:

Every fuzzy sub –bigroup of a group G is a fuzzy subgroup of the group G but not conversely

Proof:

It follows from the definition of the fuzzy sub –bigroup of a group G that every sub –bigroup of a group G is a fuzzy subgroup of the group G .

Main Theorem 1:

The union of two fuzzy subgroups of a group G is a fuzzy subgroup If and only if one is contained in the other.

Proof:

Necessary part:

Let μ_1 and μ_2 be two fuzzy subgroups of G such that one is contained in the other.

Hence either $\mu_1 \subseteq \mu_2$ (or) $\mu_2 \subseteq \mu_1$.

To prove: $\mu_1 \cup \mu_2$ is a fuzzy subgroup of G.

Let the union of two fuzzy subsets μ_1 , μ_2 is defined by

$\mu_1 \cup \mu_2\ (x) = \max\{\mu_1(x), \mu_2\ (x)\}$

So $\mu_1 \cup \mu_2(xy) = \max\{\mu_1(xy), \mu_2\ (xy)\}$……….(7)

Since μ_1 and μ_2 are the two fuzzy subgroup of G.

$\mu_1 \cup \mu_2(xy) = \mu_1(xy)$ (or)

$\mu_1 \cup \mu_2(xy) = \mu_2(xy)$…….(8)

Since $\mu_1(xy)$ and $\mu_2(xy)$ are fuzzy subgroup of G.

From (7) and (8),$\mu_1 \cup \mu_2(xy)$ is also a fuzzysubgroup of G.

Hence $\mu_1 \cup \mu_2$ is a fuzzysubgroup of G.

Sufficient part:

Suppose $\mu_1 \cup \mu_2$ is a fuzzy subgroup of G.

To Claim:$\mu_1 \subseteq \mu_2$ (or) $\mu_2 \subseteq \mu_1$.

Since μ_1, andμ_2 are fuzzy subgroup of G.

$\mu_1(xy) \geq \min\{\mu_1(x), \mu_1(y)\}$………(9)

(by condition (i) of definition of fuzzy subgroup of G).

There are two cases,

$i)\mu_1(xy) \geq \mu_1(x)$,

ii) $\mu_1(xy) \geq \mu_1(y)$.

Case: (i)

$\mu_1(xy) \geq \mu_1(x)$ … … . . (10)

By fuzzy union of fuzzy sets μ_1 and μ_2 ,we have

$\mu_1 \cup \mu_2(x) = \mu_1(x)$ … … … (11),

Sub (11) in (10), we get

$\mu_1(xy) \geq \mu_1 \cup \mu_2(x) = \mu_2(x)$.

$\mu_1(xy) \geq \mu_2(x)$ …….. (12)

From (10) and (12), we get

$\mu_1(x) \leq \mu_2(x)$.

$\mu_1 \subseteq \mu_2$…………(13)

Similarly case (ii):

$\mu_1(xy) \geq \mu_1(y)$………(14)

By definition of fuzzy union of fuzzy sets of G, we have

$\mu_1 \cup \mu_2(y) = \mu_1(y)$. … . ….(15)

Sub 9in 8we get,

$\mu_1(xy) \geq \mu_1 \cup \mu_2(y) = \mu_2(y)$.

$\mu_1(xy) \geq \mu_2(y)$….…….(16)

From (14)and (16) we have,

$\mu_2\ (y) \leq \mu_1(y)$.

$\mu_2 \subseteq \mu_1$……….(17).

From (13) and (17),

i.e $\mu_1 \subseteq \mu_2$ and $\mu_2 \subseteq \mu_1$.

Hence μ_1 is contained in μ_2 and μ_2 is contained in μ_1.

Therefore The Union of two fuzzy subgroups of a group G is a fuzzy subgroup.if and only if one contained in other.

Main Theorem 2:

The union of two fuzzy sub-bigroups of a bigroup G is a fuzzy sub-bigroup if and only if one is contained in the other.

Proof:

Let μ_1 and μ_2 are the fuzzy sub-bigroups and $\mu_1 \subseteq \mu_2$ and $\mu_2 \subseteq \mu_1$.

To prove : $\mu_1 \cup \mu_2$ is a fuzzy sub-bigroup .

By definition of Fuzzy Union of the fuzzy sets μ_1 and μ_2 (Definition : 4)

$$\mu_1 \cup \mu_2(xy) = \mu_1(xy) \text{ (or)}$$

$\mu_1 \cup \mu_2(xy) = \mu_2(xy)$.

Since $\mu_1(xy)$ and $\mu_2(xy)$ are fuzzy sub - bigroups of G.

hence , $\mu_1 \cup \mu_2$ is a fuzzy sub-bigroup .

Conversely,

Let $\mu_1 \cup \mu_2$ is a fuzzy sub-bigroup .

To prove: $\mu_1 \subseteq \mu_2$ and $\mu_2 \subseteq \mu_1$.

Since $\mu_1 \cup \mu_2$ is a fuzzy sub-bigroup .

By Theorem: 8 , every fuzzy sub –bigroup of a group G is a fuzzy subgroup of the group G but not conversely .

Hence $\mu_1 \cup \mu_2$ is a fuzzy subgroup .

and By Main Theorem : 1

The union of two fuzzy subgroups of a group G is a fuzzy subgroup if and only if one is contained in the other.

Hence $\mu_1 \subseteq \mu_2$ and $\mu_2 \subseteq \mu_1$.

Therefore, The union of two fuzzy sub-bigroups of a bigroup G is a fuzzy sub-bigroup if and only if One is contained in the other.

4. Conclusion:

In this paper, we have discussed the union of Fuzzy subgroups, properties, union of Fuzzy subgroup and proved the two Main Theorems.

5. References:

1. Arumugam.S, Isaac.A.T-Modern Algebra.
2. I. N. Herstein-Topics in Algebra.
3. Smarandache Fuzzy Algebra by W. B. Vasanthakandasamy.
4. Fuzzy Mathematical concepts by S.Nanda.N.R.Das.
5. Witoldpedrycz, Fernando Gomide-An Introduction to Fuzzy sets, Analysis, and Design.