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FEKETE-SZEGO INEQUALITIES AND $(j,k)-$ SYMMETRIC FUNCTIONS USING $q-$ DERIVATIVE

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Abstract:

In this paper sharp upper bounds of $|a_3-\mu a_2^2|$ for functions belonging to new subclasses defined using the concept of $(j,k)-$ symmetric functions using $q-$ derivative are derived. Furthermore, the application of the results are also illustrated.

Key Words: Analytic Function; Univalent Function; Schwarz Function; $q-$ Starlike, $q-$ Convex, $q-$ Derivative Operator, Subordination & Fekete-Szegö Inequality.

1. Introduction:

The $q-$ difference calculus or quantum calculus was initiated at the beginning of 19th century that was initially developed by Jackson [16, 15]. Basic definitions and properties of $q-$ deference calculus can be found in the book mentioned in [17]. The fractional $q-$ difference calculus had its origin in the works by Al-Salam [3] and Agarwal [1]. Recently, the area of $q-$ calculus has attracted the serious attention of researchers. The great interest is due to its application in various branches of mathematics and physics. Mohammed and Darus [21] studied approximation and geometric properties of these $q-$ operators for some subclasses of analytic functions in compact disk.

Let $\mathcal{A}$ denote the class of all analytic function of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1.1}$$

in the open unit disc $\mathcal{U}=\{z: z\in\mathbb{C}; |z|<1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathcal{U}$.

If f and g are analytic in $\mathcal{U}$, we say that the function f is subordinate to g, written as $f(z)\prec g(z)$ in $\mathcal{U}$, if there exist a Schwarz function $\omega(z)$, which is analytic in $\mathcal{U}$ with $w(0)=0$ and $|w(z)|<1$ such that $f(z)=g(w(z))$ for $z\in\mathcal{U}$. Furthermore, if the function $g(z)$ is univalent $\mathcal{U}$, then we have the following equivalence holds(see [9] and [20]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0) \quad and \quad f(\mathcal{U})\subset g(\mathcal{U}).$$

For function $f\in\mathcal{A}$ given by (1.1) and $0<q<1$, the $q-$ derivative of a function f is defined by (see [15, 16])

$$D_q f(z)=\frac{f(z)-f(qz)}{(1-q)z} \qquad (z\neq 0), \tag{1.2}$$

$D_q f(0)=f'(0)$ and $D_q^2 f(z)=D_q(D_q f(z))$. From (1.2), we deduce that

$$D_q f(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}, \tag{1.3}$$

where

$$[n]_q=\frac{1-q^n}{1-q}. \tag{1.4}$$

As $q\to 1^-, [n]_q\to n$. For a function $h(z)=z^n$, we observe that

$$D_q(h(z))=D_q(z^n)=\frac{1-q^n}{1-q}z^{n-1}=[n]_q z^{n-1},$$

$$\lim_{q\to 1}D_q(h(z))=\lim_{q\to 1}\left([n]_q z^{n-1}\right)=nz^{n-1}=h'(z),$$

where h' is the ordinary derivative.

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As a right inverse, Jackson [15] introduced the $q-$ integral

$$\int_0^z h(t)d_q t = z(1-q)\sum_{n=0}^{\infty} q^n f\left(zq^n\right),$$

provided that the series converges. For a function $h(z)=z^n$, we observe that

$$\int_0^z h(t)d_q t = \lim_{q\to 1^-}\frac{z^{n+1}}{[n+1]_q}=\frac{z^{n+1}}{n+1}=\int_0^z h(t)dt,$$

where $\int_0^z h(t)dt$ is the ordinary integral.

Ma and Minda [19] unified various subclasses of starlike and convex functions for which either quantity $zf'(z)/f(z)$ or quantity $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disc U, with $\phi(0)=1$, $\phi'(0)>0$, and ϕ maps U onto a region starlike, with respect to the real axis.

The class of Ma-Minda starlike functions $f(z)\in \mathrm{A}$ consists of functions satisfying the subordination $zf'(z)/f(z)\prec \phi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f(z)\in \mathrm{A}$ satisfying the subordination $1+zf''(z)/f'(z)\prec \phi(z)$.

Let k be a positive integer and $\varepsilon = exp(2\pi i/k)$. A domain D is said to be $k-$fold symmetric if a rotation of D about the origin through an angle $2\pi/k$ carries D onto itself. A function $f\in \mathrm{A}$ is said to be $k-$fold symmetric in U if for each $z\in \mathrm{U}$

$$f(\varepsilon z)=\varepsilon f(z).$$

The family of all $k-$fold symmetric functions is denoted by S^k and for $k=2$, we get class of the odd univalent functions. The notion of $(j,k)-$ symmetric functions $(k=2,3,\ldots; j=0,1,2,\ldots(k-1))$ is a generalization of even, odd, $k-$symmetrical functions. Let $\varepsilon = exp(2\pi i/k)$ and $j=0,1,2,\ldots(k-1)$ where $k\geq 2$ is a natural number. A function $f:\mathrm{U}\mapsto \mathrm{C}$ is called $(j,k)-$symmetrical if

$$f(\varepsilon z)=\varepsilon^j f(z),\ z\in \mathrm{U}.$$

We note that the family of all $(j,k)-$symmetric functions is denoted by $\mathrm{S}^{(j,k)}$. Also, $\mathrm{S}^{(0,2)}, \mathrm{S}^{(1,2)}$ and $\mathrm{S}^{(1k)}$ are called even, odd and $k-$symmetric functions respectively.
We have the following decomposition theorem (see [18]).

For every mapping $f:\mathrm{D}\mapsto \mathrm{C}$, and D is a $k-$ fold symmetric set, there exist exactly the sequence of $(j,k)-$symmetrical functions $f_{j,k}$,

$$f(z)=\frac{1}{k}\sum_{j=0}^{k-1} f_{j,k}(z), \tag{1.5}$$

where

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj} f(\varepsilon^v z), \tag{1.6}$$

$$(f\in \mathrm{A}; k=1,2,\ldots; j=0,1,2,\ldots(k-1)).$$

The decomposition (1.5) is a generalization of the well-known fact that each function defined on a symmetrical subset U of C can be uniquely represented as the sum of an even function and an odd function (see Theorem 1 of [18]). From (1.6), we can get

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj} f(\varepsilon^v z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}\left(\sum_{n=1}^{\infty} a_n(\varepsilon^v z)^n\right),$$

then

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$$f_{j,k}(z)=\sum_{n=1}^{\infty}\psi_n a_n z^n, \qquad a_1=1, \qquad \psi_n=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-j)v}=\begin{cases}1 & n=lk+j;\\ 0 & n\neq lk+j\end{cases}. \tag{1.7}$$

Definition 1.1 *(see [4]) Let* $\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$ *be univalent starlike with respect to 1 which maps the unit disk* $\mathcal{U}$ *onto a region in the right half plane which is symmetric with respect to the real axis. Let* $0\le\beta\le\alpha\le 1$ *and* $B_1>0$. *Then the function* $f(z)\in\mathcal{A}$ *is in the class* $\mathcal{S}_{j,k}(\phi)$ *if*

$$\frac{zf'(z)}{f_{j,k}(z)}\prec\phi(z).$$

Definition 1.2 *A function* $f\in\mathcal{A}$ *is said to be in the class* $\mathcal{S}^q_{j,k}(\phi)$ *if it satisfies the following subordination condition:*

$$\frac{zD_qf(z)}{f_{j,k}(z)}\prec\phi(z) \qquad (\phi\in\mathcal{P}). \tag{1.8}$$

Definition 1.3 *A function* $f\in\mathcal{A}$ *is said to be in the class* $\mathcal{C}^q_{j,k}(\phi)$, *if it satisfies the following subordination condition:*

$$\frac{D_q(zD_qf(z))}{D_qf_{j,k}(z)}\prec\phi(z) \qquad (\phi\in\mathcal{P}). \tag{1.9}$$

Lemma 1 *[19] Let* $p(z)\in\mathcal{P}$ *and also let* v *be a complex number, then*

$$|c_2-vc_1^2|\le 2\max\{1,|2v-1|\},$$

the result is sharp for functions given by

$$p(z)=\frac{1+z^2}{1-z^2}, \quad p(z)=\frac{1+z}{1-z}.$$

Lemma 2 *[19] Let* $p(z)\in\mathcal{P}$, *then*

$$|c_2-vc_1^2|\le\begin{cases}-4v+2, & \text{if} \quad v\le 0;\\ 2, & \text{if} \quad 0\le v\le 1;\\ 4v-2, & \text{if} \quad v\ge 1.\end{cases} \tag{1.10}$$

When $v<0$ or $v>1$, the equality holds if and only if $p(z)=(1+z)/(1-z)$ or one of its rotations. If $0<v<1$, then the equality if and only if $p(z)=(1+z^2)/(1-z^2)$ or one of its rotations. If $v=0$, the equality holds if and only if

$$p(z)=\left(\frac{1}{2}+\frac{1}{2}\vartheta\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\vartheta\right)\frac{1-z}{1+z},(0\le\vartheta\le 1),$$

or one of its rotations. If $v=1$, the equality holds if and only if

$$\frac{1}{p(z)}=\left(\frac{1}{2}+\frac{1}{2}\vartheta\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\vartheta\right)\frac{1-z}{1+z},(0\le\vartheta\le 1).$$

Also the above upper bound is sharp and it can be improved as follows when $0\le v\le 1$

$$|c_2-vc_1^2|+v|c_1|^2\le 2, \qquad (0<v\le 1/2),$$
$$|c_2-vc_1^2|+(1-v)|c_1|^2\le 2, \qquad (1/2\le v<1).$$

In the present paper, we obtain the Fekete-Szegö inequalities for the class $\mathcal{S}^q_{j,k}(\phi)$ and $\mathcal{C}^q_{j,k}(\phi)$. We employ the technique adapted by Ma and Minda [19] to find the coefficient estimates for our class.

2. Main Results:

Unless otherwise mentioned, we assume throughout this paper that the function $0<q<1, \phi\in\mathcal{P}, [n]_q$ is given by (1.4) and $z\in\mathcal{U}$.

Theorem 1 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$ *and* $B_2\ge 0$. *Let*

$$\sigma_1 = \frac{\left([2]_q - \psi_2\right)B_1^2\psi_2 + \left([2]_q - \psi_2\right)^2(B_2 - B_1)}{\left([3]_q - \psi_3\right)B_1^2}, \tag{2.1}$$

$$\sigma_2 = \frac{\left([2]_q - \psi_2\right)B_1^2\psi_2 + \left([2]_q - \psi_2\right)^2(B_2 + B_1)}{\left([3]_q - \psi_3\right)B_1^2}. \tag{2.2}$$

If $f(z)$ given by (1.1) belongs to $S_{j,k}^q(\phi)$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{B_2}{[3]_q - \psi_3} + \dfrac{B_1^2}{[2]_q - \psi_2}\left(\dfrac{\psi_2}{[3]_q - \psi_3} - \dfrac{\mu}{[2]_q - \psi_2}\right) & \text{if} \quad \mu \le \sigma_1, \\ \dfrac{B_1}{[3]_q - \psi_3} & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ -\dfrac{B_2}{[3]_q - \psi_3} - \dfrac{B_1^2}{[2]_q - \psi_2}\left(\dfrac{\psi_2}{[3]_q - \psi_3} - \dfrac{\mu}{[2]_q - \psi_2}\right) & \text{if} \quad \mu \ge \sigma_2. \end{cases} \tag{2.3}$$

where ψ_n is defined by (1.7). The result is sharp.

Proof. If $f \in S_{j,k}^q(\phi)$, then there exists a Schwarz function $\omega(z)$, which is analytic in U with $w(0)=0$ and $|w(z)| < 1 \in U$ such that

$$\frac{zD_q f(z)}{f_{j,k}(z)} = \phi(\omega(z)). \tag{2.4}$$

Define the function $p(z)$ by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots, z \in U. \tag{2.5}$$

Since $\omega(z)$ is Schwarz function, we see that $Re\, p(z) > 0$ and $p(z) = 1$.
Therefore

$$\begin{aligned} \phi(\omega(z)) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right) \\ &= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right]\right) \\ &= 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \cdots. \end{aligned} \tag{2.6}$$

Now by substituting (2.6) in (2.4), we have

$$\frac{zD_q f(z)}{f_{j,k}(z)} = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \cdots.$$

From this equation, we obtain

$$([2]_q - \psi_2)a_2 = \frac{B_1 c_1}{2}$$

$$([3]_q - \psi_3)a_3 - \left([2]_q - \psi_2\right)\psi_2 a_2^2 = \frac{B_1 c_2}{2} - \frac{B_1 c_1^2}{4} + \frac{B_2 c_1^2}{4},$$

or equivalently

$$a_2 = \frac{B_1 c_1}{2\left([2]_q - \psi_2\right)}$$

$$a_3 = \frac{B_1}{2([3]_q - \psi_3)}\left(c_2 - \frac{c_1^2}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1\psi_2}{[2]_q - \psi_2}\right)\right).$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{B_1}{2([3]_q - \psi_3)}(c_2 - vc_1^2), \tag{2.7}$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{B_1}{[2]_q - \psi_2}\left(\psi_2 - \frac{[3]_q - \psi_3}{[2]_q - \psi_2}\mu\right)\right]. \tag{2.8}$$

Our result now follows by an application of Lemma 2.

To show that the bounds are sharp, we define the functions $K_{\phi n}(n = 2,3,4\ldots)$ by

$$\frac{zD_q K_{\phi n}(z)}{K_{\phi n}(z)} = \phi(z^{n-1}), \qquad K_{\phi n}(0) = 0 = K'_{\phi n}(0) - 1$$

and the functions F_λ and $G_\lambda(0 \le \lambda \le 1)$ by

$$\frac{zD_q F_\lambda(z)}{F_\lambda(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \qquad F_\lambda(0) = 0 = F'_\lambda(0) - 1$$

and

$$\frac{zD_q G_\lambda(z)}{G_\lambda(z)} = \phi\left(-\frac{1+\lambda z}{z(z+\lambda)}\right), \qquad G_\lambda(0) = 0 = G'_\lambda(0) - 1.$$

Clearly, the functions $K_{\phi n}, F_\lambda$ and $G_\lambda \in S^q_{j,k}(\phi)$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$, or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$, or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_λ, or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_λ, or one of its rotations.

Theorem 2 *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. Let $f(z)$ given by (1.1) belongs to $S^q_{j,k}(\phi)$ and σ_3 given by*

$$\sigma_3 = \frac{([2]_q - \psi_2)B_1^2\psi_2 + ([2]_q - \psi_2)^2 B_2}{([3]_q - \psi_3)B_1^2}.$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_q - \psi_2)^2}{([3]_q - \psi_3)B_1^2}\left[B_1 - B_2 - \frac{B_1^2}{[2]_q - \psi_2}\left(\psi_2 - \frac{[3]_q - \psi_3}{[2]_q - \psi_2}\mu\right)\right]|a_2|^2 \le \frac{B_1}{[3]_q - \psi_3}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_q - \psi_2)^2}{([3]_q - \psi_3)B_1^2}\left[B_1 + B_2 + \frac{B_1^2}{[2]_q - \psi_2}\left(\psi_2 - \frac{[3]_q - \psi_3}{[2]_q - \psi_2}\mu\right)\right]|a_2|^2 \le \frac{B_1}{[3]_q - \psi_3},$$

where ψ_n is defined by (1.7). The result is sharp.

Remark 1

- For $q \to 1^-$ in Theorem 1 and 2, we get the result similar to those obtained by Al Sarari and Latha in [4].
- For $q \to 1^-, j = 1$ in Theorem 1 and 2, we get the result similar to those obtained by Al-Shaqsi and Darus in [5].
- For $q \to 1^-, j = 1, k = 2$ in Theorem 1 and 2, we get the result similar to those obtained by Shanmugam et al. in [28].
- For $q \to 1^-, j = 1, k = 1$ in Theorem 1 and 2, we get the result similar to those obtained by Ma and Minda in [19].

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Theorem 3 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots(B_1\neq 0)$. *If* $f(z)\in S^q_{j,k}(\phi)$, *then*

$$|a_3-\mu a_2^2|\leq \frac{|B_1|}{[3]_q-\psi_3}\max\left\{1;\left|\frac{B_2}{B_1}+\frac{B_1}{[2]_q-\psi_2}\left(\psi_2-\frac{[3]_q-\psi_3}{[2]_q-\psi_2}\mu\right)\right|\right\}. \quad (2.9)$$

The result is sharp.

Taking $q\to 1^-$ in Theorem 3, we obtain the following result for the functions belonging to the class $S_{j,k}(\phi)$.

Corollary 1 *[4] Let* $\phi(z)=1+B_1z+B_2z^2+\cdots(B_1\neq 0)$. *If* $f(z)\in S_{j,k}(\phi)$, *then*

$$|a_3-\mu a_2^2|\leq \frac{|B_1|}{3-\psi_3}\max\left\{1;\left|\frac{B_2}{B_1}+\frac{B_1}{2-\psi_2}\left(\psi_2-\frac{3-\psi_3}{2-\psi_2}\mu\right)\right|\right\}.$$

The result is sharp.

Taking $q\to 1^-, j=1$ and $k=1$ in Theorem 3, we obtain the following result for the functions belonging to the class $S_{1,1}(\phi)$.

Corollary 2 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots(B_1\neq 0)$. *If* $f(z)\in S_{1,1}(\phi)$, *then*

$$|a_3-\mu a_2^2|\leq \frac{|B_1|}{2}\max\left\{1;\left|\frac{B_2}{B_1}+B_1(1-2\mu)\right|\right\}.$$

The result is sharp.

Taking $q\to 1^-, j=1$ and $k=2$ in Theorem 3, we obtain the following result for the functions belonging to the class $S_{1,2}(\phi)$.

Corollary 3 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots(B_1\neq 0)$. *If* $f(z)\in S_{1,2}(\phi)$, *then*

$$|a_3-\mu a_2^2|\leq \frac{|B_1|}{2}\max\left\{1;\left|\frac{B_2}{B_1}-\frac{B_1\mu}{2}\right|\right\}.$$

The result is sharp.

Theorem 4 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$. *If* $f(z)$ *given by (1.1) belongs to* $C^q_{j,k}(\phi)$, *then*

$$|a_3-\mu a_2^2|\leq \frac{|B_1|}{[3]_q([3]_q-\psi_3)}\max\left\{1;\left|\frac{B_2}{B_1}+\frac{B_1}{[2]_q-\psi_2}\left(\psi_2-\frac{[3]_q([3]_q-\psi_3)}{([2]_q)^2([2]_q-\psi_2)}\mu\right)\right|\right\}. \quad (2.10)$$

The result is sharp.

Taking $q\to 1^-$ in Theorem 4, we obtain the following result for the functions belonging to the class $C_{j,k}(\phi)$.

Corollary 4 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$. *If* $f(z)$ *given by (1.1) belongs to* $C_{j,k}(\phi)$, *then*

$$|a_3-\mu a_2^2|\leq \frac{|B_1|}{3(3-\psi_3)}\max\left\{1;\left|\frac{B_2}{B_1}+\frac{B_1}{2-\psi_2}\left(\psi_2-\frac{3(3-\psi_3)}{2^2(2-\psi_2)}\mu\right)\right|\right\}.$$

The result is sharp.

Taking $q\to 1^-, j=1$ and $k=1$ in Theorem 4, we obtain the following result for the functions belonging to the class $C_{1,1}(\phi)$.

Corollary 5 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$. *If* $f(z)$ *given by (1.1) belongs to* $C_{1,1}(\phi)$, *then*

$$|a_3-\mu a_2^2|\leq \frac{|B_1|}{6}\max\left\{1;\left|\frac{B_2}{B_1}+B_1\left(1-\frac{3\mu}{2}\right)\right|\right\}.$$

The result is sharp.

Taking $q\to 1^-, j=1$ and $k=2$ in Theorem 4, we obtain the following result for the functions belonging to the class $\mathcal{C}_{1,2}(\phi)$.

Corollary 6 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$. *If* $f(z)$ *given by (1.1) belongs to* $\mathcal{C}_{1,2}(\phi)$, *then*

$$|a_3-\mu a_2^2|\leq\frac{|B_1|}{6}max\left\{1;\left|\frac{B_2}{B_1}-\frac{3B_1\mu}{8}\right|\right\}.$$

The result is sharp.

Theorem 5 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$ *and* $B_2\geq 0$. *Let*

$$\chi_1=\frac{([2]_q)^2([2]_q-\psi_2)\left[B_1^2\psi_2+([2]_q-\psi_2)(B_2-B_1)\right]}{B_1^2[3]_q([3]_q-\psi_3)},$$

$$\chi_2=\frac{([2]_q)^2([2]_q-\psi_2)\left[B_1^2\psi_2+([2]_q-\psi_2)(B_2+B_1)\right]}{B_1^2[3]_q([3]_q-\psi_3)},$$

$$\chi_3=\frac{([2]_q)^2([2]_q-\psi_2)\left[B_1^2\psi_2+([2]_q-\psi_2)B_2\right]}{B_1^2[3]_q([3]_q-\psi_3)}.$$

If $f(z)$ given by (1.1) belongs to $\mathcal{C}_{j,k}^q(\phi)$ with $b>0$, then

$$|a_3-\mu a_2^2|\leq\begin{cases}\dfrac{B_2}{[3]_q([3]_q-\psi_3)}+\dfrac{B_1^2}{[3]_q([3]_q-\psi_3)([2]_q-\psi_2)}\left(\psi_2-\dfrac{[3]_q([3]_q-\psi_3)}{([2]_q)^2([2]_q-\psi_2)}\mu\right) & if \quad \mu\leq\chi_1,\\ \dfrac{B_1}{[3]_q([3]_q-\psi_3)} & if \quad \chi_1\leq\mu\leq\chi_2,\\ -\dfrac{B_2}{[3]_q([3]_q-\psi_3)}-\dfrac{B_1^2}{[3]_q([3]_q-\psi_3)([2]_q-\psi_2)}\left(\psi_2-\dfrac{[3]_q([3]_q-\psi_3)}{([2]_q)^2([2]_q-\psi_2)}\mu\right) & if \quad \mu\geq\chi_2.\end{cases}$$

(2.11)

Further, if $\chi_1\leq\mu\leq\chi_3$, then

$$|a_3-\mu a_2^2|+\frac{([2]_q)^2([2]_q-\psi_2)^2}{[3]_q([3]_q-\psi_3)B_1^2}\left[B_1-B_2-\frac{B_1^2}{[2]_q-\psi_2}\left(\psi_2-\frac{[3]_q([3]_q-\psi_3)}{([2]_q)^2([2]_q-\psi_2)}\mu\right)\right]|a_2|^2$$
$$\leq\frac{B_1}{[3]_q([3]_q-\psi_3)}.$$

If $\chi_3\leq\mu\leq\chi_2$, then

$$|a_3-\mu a_2^2|+\frac{([2]_q)^2([2]_q-\psi_2)^2}{[3]_q([3]_q-\psi_3)B_1^2}\left[B_1+B_2+\frac{B_1^2}{[2]_q-\psi_2}\left(\psi_2-\frac{[3]_q([3]_q-\psi_3)}{([2]_q)^2([2]_q-\psi_2)}\mu\right)\right]|a_2|^2$$
$$\leq\frac{B_1}{[3]_q([3]_q-\psi_3)}.$$

The result is sharp.

Taking $q\to 1^-$ in Theorem 5, we obtain the following result for the functions belonging to the class $\mathcal{C}_{j,k}(\phi)$.

Corollary 7 *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$ *and* $B_2\geq 0$. *Let*

$$\chi_1=\frac{(2)^2(2-\psi_2)\left[B_1^2\psi_2+(2-\psi_2)(B_2-B_1)\right]}{B_1^2 3(3-\psi_3)},$$

$$\chi_2 = \frac{(2)^2(2-\psi_2)\left[B_1^2\psi_2+(2-\psi_2)(B_2+B_1)\right]}{B_1^2 3(3-\psi_3)},$$

$$\chi_3 = \frac{(2)^2(2-\psi_2)\left[B_1^2\psi_2+(2-\psi_2)B_2\right]}{B_1^2 3(3-\psi_3)}.$$

If $f(z)$ given by (1.1) belongs to $C_{j,k}(\phi)$ with $b>0$, then

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{B_2}{3(3-\psi_3)}+\dfrac{B_1^2}{3(3-\psi_3)(2-\psi_2)}\left(\psi_2-\dfrac{3(3-\psi_3)}{(2)^2(2-\psi_2)}\mu\right) & if \quad \mu\le\chi_1, \\ \dfrac{B_1}{3(3-\psi_3)} & if \quad \chi_1\le\mu\le\chi_2, \\ -\dfrac{B_2}{3(3-\psi_3)}-\dfrac{B_1^2}{3(3-\psi_3)(2-\psi_2)}\left(\psi_2-\dfrac{3(3-\psi_3)}{(2)^2(2-\psi_2)}\mu\right) & if \quad \mu\ge\chi_2. \end{cases}$$

Further, if $\chi_1\le\mu\le\chi_3$, then

$$|a_3-\mu a_2^2|+\frac{(2)^2(2-\psi_2)^2}{3(3-\psi_3)B_1^2}\left[B_1-B_2-\frac{B_1^2}{2-\psi_2}\left(\psi_2-\frac{3(3-\psi_3)}{(2)^2(2-\psi_2)}\mu\right)\right]|a_2|^2\le\frac{B_1}{3(3-\psi_3)}.$$

If $\chi_3\le\mu\le\chi_2$, then

$$|a_3-\mu a_2^2|+\frac{(2)^2(2-\psi_2)^2}{3(3-\psi_3)B_1^2}\left[B_1+B_2+\frac{B_1^2}{2-\psi_2}\left(\psi_2-\frac{3(3-\psi_3)}{(2)^2(2-\psi_2)}\mu\right)\right]|a_2|^2\le\frac{B_1}{3(3-\psi_3)}.$$

The result is sharp.

Remark 2

- For $q\to1^-, j=1, k=1$ in Theorem 5, we get the result similar to those obtained by Ma and Minda in [19].
- For $q\to1^-, j=1, k=2$ in Theorem 5, we get the result similar to those obtained by Shanmugam et al. in [28].

3. Applications to Functions Defined by Fractional Derivatives:

In order to introduce classes $S_{j,k}^{q,\lambda}(\phi)$ and $C_{j,k}^{q,\lambda}(\phi)$ we need the following.

Definition 3.1 *Let $f(z)$ be analytic in a simply connected region of the $Z-$plane containing the origin. The fractional derivative of f of order λ is defined by*

$$D_z^\lambda f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^\lambda}d\zeta,\quad (0\le\lambda<1) \tag{3.1}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring that $log(z-\zeta)$ to be real for $z-\zeta>0$.

Using the definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [22] introduced the operator $\Omega^\lambda : A\to A$ defined by

$$(\Omega^\lambda f)(z)=\Gamma(2-\lambda)z^\lambda D_z^\lambda f(z),\quad (\lambda\ne2,3,4,\ldots). \tag{3.2}$$

Classes $S_{j,k}^{q,\lambda}(\phi)$ and $C_{j,k}^{q,\lambda}(\phi)$ consist of functions $f\in A$ for which $\Omega^\lambda f\in S_{j,k}^q(\phi)$ and $\Omega^\lambda f\in C_{j,k}^q(\phi)$, respectively.

For a fixed $g\in A$, let $S_{j,k}^{q,g}(\phi)$ be the class of functions $f\in A$ for which $(f*g)\in S_{j,k}^q(\phi)$ and let $C_{j,k}^{q,g}(\phi)$ be the class of functions $f\in A$ for which $(f*g)\in C_{j,k}^q(\phi)$.

Classes $S_{j,k}^{q,\lambda}(\phi)$ and $C_{j,k}^{q,\lambda}(\phi)$ are the special case of classes $S_{j,k}^{q,g}(\phi)$ and $C_{j,k}^{q,g}(\phi)$, respectively, when

$$g(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n z^n. \tag{3.3}$$

Let

$$g(z)=z+\sum_{n=2}^{\infty}g_n z^n \quad (g_n>0),$$
$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \tag{3.4}$$

Since $f\in S_{j,k}^{q,g}(\phi)(C_{j,k}^{q,g}(\phi))$ if and only if $f*g\in S_{j,k}^{q}(\phi)(C_{j,k}^{q}(\phi))$, we obtain the coefficient estimates for functions in classes $S_{j,k}^{q,g}(\phi)$ and $C_{j,k}^{q,g}(\phi)$, from the corresponding estimates for functions in classes $S_{j,k}^{q}(\phi)$ and $C_{j,k}^{q}(\phi)$.

Applying Theorem 1 for the function $(f*g)(z)=z+g_2a_2z^2+g_3a_3z^3+\cdots$, we get the following theorem for an obvious change of μ.

Theorem 6 *Let the function* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *and let* $g(z)=z+\sum_{n=2}^{\infty}g_nz^n\ (g_n>0)$. *If* $f(z)$ *given by(1.3) belongs to* $S_{j,k}^{q,g}(\phi)$, *then*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{1}{g_3([3]_q-\psi_3)}\left[B_2+\dfrac{B_1^2\psi_2}{[2]_q-\psi_2}-\mu\dfrac{g_3}{g_2^2}\dfrac{B_1^2([3]_q-\psi_3)}{([2]_q-\psi_2)^2}\right], & if\ \ \mu\le\tau_1;\\ \dfrac{B_1}{g_3([3]_q-\psi_3)}, & if\ \ \tau_1\le\mu\le\tau_2;\ \text{where}\\ -\dfrac{1}{g_3([3]_q-\psi_3)}\left[B_2+\dfrac{B_1^2\psi_2}{[2]_q-\psi_2}-\mu\dfrac{g_3}{g_2^2}\dfrac{B_1^2([3]_q-\psi_3)}{([2]_q-\psi_2)^2}\right], & if\ \ \mu\ge\tau_2.\end{cases}$$

$$\tau_1=\frac{g_2^2([2]_q-\psi_2)^2}{g_3B_1([3]_q-\psi_3)}\left[-1+\frac{B_2}{B_1}+\frac{B_1\psi_2}{([2]_q-\psi_2)}\right],$$

and

$$\tau_2=\frac{g_2^2([2]_q-\psi_2)^2}{g_3B_1([3]_q-\psi_3)}\left[1+\frac{B_2}{B_1}+\frac{B_1\psi_2}{([2]_q-\psi_2)}\right],$$

where ψ_n is defined by (1.7). The result is sharp.

Remark 3 *Since* $\Omega^{\lambda}f(z)=z+\sum_{n=2}^{\infty}\dfrac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_nz^n$,

we have

$$g_2=\frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)}=\frac{2}{2-\lambda}, \tag{3.5}$$

$$g_3=\frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)}=\frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.6}$$

For g_2,g_3 given by (3.5) and (3.6) respectively, Theorem 6 reduces to the following Corollary.

Corollary 8 *Let* $\lambda<2$. *If* $f(z)$ *given by(1.3) belongs to* $S_{j,k}^{q,\lambda}(\phi)$, *then*

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{(2-\lambda)(3-\lambda)}{6([3]_q-\psi_3)}\left[B_2+\dfrac{B_1^2\psi_2}{[2]_q-\psi_2}-\mu\dfrac{3(2-\lambda)}{2(3-\lambda)}\dfrac{B_1^2([3]_q-\psi_3)}{([2]_q-\psi_2)^2}\right], & if\ \ \mu\le\tau_1^*; \\ \dfrac{(2-\lambda)(3-\lambda)B_1}{6([3]_q-\psi_3)}, & if\ \ \tau_1^*\le\mu\le\tau_2^*; \\ -\dfrac{(2-\lambda)(3-\lambda)}{6([3]_q-\psi_3)}\left[B_2+\dfrac{B_1^2\psi_2}{[2]_q-\psi_2}-\mu\dfrac{3(2-\lambda)}{2(3-\lambda)}\dfrac{B_1^2([3]_q-\psi_3)}{([2]_q-\psi_2)^2}\right], & if\ \ \mu\ge\tau_2^*. \end{cases}$$

where

$$\tau_1^* = \frac{2(3-\lambda)}{3(2-\lambda)}\frac{([2]_q-\psi_2)^2}{B_1([3]_q-\psi_3)}\left[-1+\frac{B_2}{B_1}+\frac{B_1\psi_2}{([2]_q-\psi_2)}\right],$$

and

$$\tau_2^* = \frac{2(3-\lambda)}{3(2-\lambda)}\frac{([2]_q-\psi_2)^2}{B_1([3]_q-\psi_3)}\left[1+\frac{B_2}{B_1}+\frac{B_1\psi_2}{([2]_q-\psi_2)}\right],$$

where ψ_n is defined by (1.7). The result is sharp.

Remark 4

- For $q\to 1^-$ in Theorem 6 and Corollary 8, we get the result similar to those obtained by Al Sarari and Latha in [4].
- For $q\to 1^-, j=1$ in Theorem 6, we get the result similar to those obtained by Al-Shaqsi and Darus in [5].
- For $q\to 1^-, j=1, k=2$ in Theorem 6, we get the result similar to those obtained by Shanmugam et al. in [28].

Theorem 7 *Let the function* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *and let* $g(z)=z+\sum_{n=2}^{\infty}g_nz^n\ (g_n>0)$. *If* $f(z)$ *given by(1.3) belongs to* $\mathcal{C}_{j,k}^{q,g}(\phi)$, *then*

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{1}{g_3[3]_q([3]_q-\psi_3)}\left[B_2+\dfrac{B_1^2\psi_2}{[2]_q-\psi_2}-\mu\dfrac{g_3}{g_2^2}\dfrac{B_1^2[3]_q([3]_q-\psi_3)}{[2]_q^2([2]_q-\psi_2)^2}\right], & if\ \ \mu\le\tau_1; \\ \dfrac{B_1}{g_3[3]_q([3]_q-\psi_3)}, & if\ \ \tau_1\le\mu\le\tau_2; \\ -\dfrac{1}{g_3[3]_q([3]_q-\psi_3)}\left[B_2+\dfrac{B_1^2\psi_2}{[2]_q-\psi_2}-\mu\dfrac{g_3}{g_2^2}\dfrac{B_1^2[3]_q([3]_q-\psi_3)}{[2]_q^2([2]_q-\psi_2)^2}\right], & if\ \ \mu\ge\tau_2. \end{cases}$$

where

$$\tau_1 = \frac{g_2^2[2]_q^2([2]_q-\psi_2)^2}{g_3B_1[3]_q([3]_q-\psi_3)}\left[-1+\frac{B_2}{B_1}+\frac{B_1\psi_2}{([2]_q-\psi_2)}\right],$$

and

$$\tau_2 = \frac{g_2^2[2]_q^2([2]_q-\psi_2)^2}{g_3B_1[3]_q([3]_q-\psi_3)}\left[1+\frac{B_2}{B_1}+\frac{B_1\psi_2}{([2]_q-\psi_2)}\right],$$

where ψ_n is defined by (1.7). The result is sharp.

For g_2, g_3 given by (3.5) and (3.6) respectively, Theorem 7 reduces to the following Corollary.

Corollary 9 *Let* $\lambda<2$. *If* $f(z)$ *given by(1.3) belongs to* $\mathcal{C}_{j,k}^{q,\lambda}(\phi)$, *then*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{(2-\lambda)(3-\lambda)}{6[3]_q([3]_q-\psi_3)}\left[B_2+\dfrac{B_1^2\psi_2}{[2]_q-\psi_2}-\mu\dfrac{3(2-\lambda)}{2(3-\lambda)}\dfrac{B_1^2[3]_q([3]_q-\psi_3)}{[2]_q^2([2]_q-\psi_2)^2}\right], & if\ \ \mu\le\tau_1^*;\\ \dfrac{(2-\lambda)(3-\lambda)B_1}{6[3]_q([3]_q-\psi_3)}, & if\ \ \tau_1^*\le\mu\le\tau_2^*;\\ -\dfrac{(2-\lambda)(3-\lambda)}{6[3]_q([3]_q-\psi_3)}\left[B_2+\dfrac{B_1^2\psi_2}{[2]_q-\psi_2}-\mu\dfrac{3(2-\lambda)}{2(3-\lambda)}\dfrac{B_1^2[3]_q([3]_q-\psi_3)}{[2]_q^2([2]_q-\psi_2)^2}\right], & if\ \ \mu\ge\tau_2^*.\end{cases}$$

where

$$\tau_1^*=\frac{2(3-\lambda)}{3(2-\lambda)}\frac{[2]_q^2([2]_q-\psi_2)^2}{B_1[3]_q([3]_q-\psi_3)}\left[-1+\frac{B_2}{B_1}+\frac{B_1\psi_2}{([2]_q-\psi_2)}\right],$$

and

$$\tau_2^*=\frac{2(3-\lambda)}{3(2-\lambda)}\frac{[2]_q^2([2]_q-\psi_2)^2}{B_1[3]_q([3]_q-\psi_3)}\left[1+\frac{B_2}{B_1}+\frac{B_1\psi_2}{([2]_q-\psi_2)}\right],$$

where ψ_n is defined by (1.7). The result is sharp.

Remark 5:

For $q\to 1^-, j=1, k=2$ in Theorem 7 and Corollary 9, we get the result similar to those obtained by Shanmugam et al. in [28].

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