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A STUDY ON REGULARITY OF BLOCK TRIANGULAR FUZZY MATRIX

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Abstract:

This paper aims to assist social scientists to analyze their problems using fuzzy models. The basic and essential fuzzy matrix theory is given. The paper does not promise to give the complete properties of basic fuzzy theory or basic fuzzy matrices. Instead, the authors have only tried to give those essential basically needed to develop, the fuzzy model. The authors do not present elaborate mathematical theories to work with fuzzy matrices. Instead they have given only the needed properties by way of examples. The authors feel that the paper should mainly help social scientists, who are interested in finding out ways to emancipatae the society. Every thing is kept at simplest level and even difficult definitions, have been omitted. Another main feature of this paper is the decription of each fuzzy model using examples from real-word problems. Further. This paper gives lots of reference so that the interested reader can make use of them.

Introduction:

This paper has two sections, in section one, basic concepts about fuzzy matrices are introduced. Basic notations of matrices are given in section one in order to make the book self-contained. Section two gives the properties of fuzzy matrices. Since the data need to be transformed into fuzzy models, some elementary properties of graphs are given. Further, this section provides details of how to prepare in linguistic question to make use of in these fuzzy models when the data related with the problem is unsupervised. Fuzzy matrix theory is a mathematical theory developed by D.r.w.B.Vasantha kandasamy, k.llantheral and others. The basic concept of the theory is that mathematical matrices can be applied to social and natural situations to predict like outcomes. The mathematics in fuzzy matrix theory is very simple can be used to analyze a broad range of data. This theory is a branch of matrix theory which uses algorithms and algebra to analyze data. It is used by social scientists to analyze interactions between actors, and can be used to complement analyses carried out using game theory or other analytical tools.

Fuzzy Sets: If X is a collection of objects denoted generally by x, then a Fuzzy set A in X is a set of ordered pairs. $A=\{(x,\mu_A(x))/x\in X\}$, $\mu_A(x)$ Is called the membership of x in A which maps X to I= [0, 1].

Support: For every fuzzy subset support of A is defined by $\sup(A)=\{x\in X, A(x)>0\}$ The support of a fuzzy set A within a universal set X is the crisp set that contains all the elements of X that have none zero membership grades in A. The support of A is same as the strong α -cut of A for α =0.

Crisp Set: A fuzzy set $A\in I^*$ is called a crisp subset of X, if there is an ordinary subset $u\subset x$ such that $A=\chi u: u\to\{0,d\}\subset I$.

Fuzzy Algebra: The Boolean algebra [0, 1] is said to be the fuzzy algebra where for any two elements $x,y\in[0,1]$ we define, L.u.b by l.u.b {x, y} =x$\vee$y=max{x, y} and G.l.b by g.l.b {x, y} =x$\wedge$y=min{x, y}.

Fuzzy Matrix: Fuzzy matrix means a matrix whose entries are in [0, 1].

Pseudo-Metric: By a pseudo-metric in a set X, we mean a real valued function d: X×X$\rightarrow$R. Satisfying the following inequality conditions.

- ✓ The triangular inequality: For any three points a, b, c in X. We have
$$d(a,c)+d(b,c)\geq d(a,b)$$
- ✓ For every point x in X we have $d(x,x)=0$.
- ✓ $d(a,b)\geq 0$.
- ✓ $d(a,b)=d(b,a)$ For any two point a and b in X.

Metric: By a metric in a set X we mean a pseudo metric $d: X^2 \rightarrow R$ in X which satisfies the following condition for any two points a and b in X. d(a, b)=0 $\rightarrow$ a = b.

Semi-Ring: A semi ring in a 'S' together with two binary Operations S (+,*) satisfying the following conditions.

Additive Associative: a+(b+c) = (a+b)+c.
Additive Commutative: a+b = b+a.
Multiplicative Associative: a*(b*c) = (a*b)*c & (b*c)*a = b *(c*a).
Left and Right Distributive: a*(b+c)=(a*b)+(a*c) & (b+c)*a= (b*a) +(c*a).

Decomposition of Fuzzy Matrix:

Some operations and notation are defined, For x, y in the interval [0, 1], x+y, xy, x-y, x*y are defined as follows. x+y = max(x, y), xy = min(x, y)

$$x-y=\begin{Bmatrix} x \text{ if } x>y \\ 0 \text{ if } x\leq y \end{Bmatrix}$$

$$x*y=\begin{Bmatrix} 1 \text{ if } x\geq y \\ x \text{ if } x<y \end{Bmatrix}$$

Next we define some matrix operations on fuzzy matrices whose elements exists in the interval [0, 1]. Let

$$A=[a_{ij}]\ (m\times n)$$
$$B=[b_{ij}]\ (m\times n)\ .$$

$F=[f_{ij}],\ (n\times l)$, and $R=(r_{ij}),\ (n\times n)$ Then the following operations are defined.

$$AF=\left[\sum_{k=1}^{n} a_{ik} f_{kj}\right]$$

$$A*F=\left[\prod_{k=1}^{n}(a_{ik}*f_{kj})\right]$$

$A'=[a_{ij}]$ (The transpose of A) $A\leq B$ If and only if $a_{ij}<b_{ij}$ for every i, j. $\Delta R=R-R'$.

Transitive: A matrix R is said to be transitive if $R^2\leq R$

Reflexive: A matrix R all of whose diagonal elements are one is called reflexive.

Irreflexive: Conversely a matrix R all of whose diagonal elements are zero is called irreflexive.

Nilpotent: A matrix R is nilpotent if $R^n=0$ (0 is the zero matrixes) we deal only with fuzzy matrices.

Lemma:

If $A=[a_{ij}]$ is a $m\times n$ fuzzy matrix then $A*A'$ is reflexive and transitive.

Proof:

Let $S=[S_{ij}]=A*A^{'}$.

$$(ie)\quad S_{ij}=\prod_{k=1}^{n}(a_{ik}*a_{jk})=1$$

Clearly, $$S_{ii}=\prod_{k=1}^{n}(a_{ik}*a_{jk})=1.$$

Thus S is reflexive. Suppose that $S_{il}S_{lj}=c>0$ For some l, then

$$S_{il}=\prod_{k=1}^{n}(a_{ik}*a_{lk})\geq c$$

$$S_{ij}=\prod_{k=1}^{n}(a_{ik}*a_{jk})\geq c.$$

If $S_{ij}<c$ then $a_{ih}<a_{jh}$ and $a_{ih}<c$ for some h. Therefore $\because S_{il}\geq c$

And $S_{lj}\geq c$ we have $c>q_{lh}\geq a_{lh}\geq a_{j}$ Which is the contradiction. Hence $S_{ij}\geq c$ so that S is transitive. Let A_i be the i^{th} row of A if $A_i\geq A_j$ then $S_{ij}=1$ where S_{ij} is the (i, j) entry of $S=A*A^{'}$. Hence the matrix S represents inclusion among the rows of A. In other words S gives the hierarchy is reflexive and transitive. This becomes clear if A is Boolean. Interesting properties. If a $n\times n$ fuzzy matrix R is reflexive and transitive then as is well-known R is idempotent that is $R^2=R$.

Lemma:

Let $S=[S_{ij}]$ and $Q=[q_{ij}]$ be $m\times m$ transitive matrices. If $S\leq Q$ then $S-Q^{'}$ is reflexive and transitive.

Proof:

Let $H=[h_{ij}]=S-Q^{'}$ that is $h_{ij}=S_{ij}-q_{ji}$ then $h_{ii}=S_{ii}-q_{ii}=0$. So that H is irreflexive next suppose that $h_{ik}h_{kj}=c>0$ then there are two cases.

Case (i): $S_{ik}=C$, $S_{ik}>Q_{ki}$, $S_{kj}\geq C$

Case (ii): $S_{ik}\geq C$, $S_{kj}=C$, $S_{kj}>q_{jk}$ Clearly $S_{ij}\geq C$ suppose that $q_{ij}\geq c$ in the first cases $q_{jk}\geq q_{ji}q_{ik}\geq c$. This is contradiction. Hence $q_{ji}<c$ so that $h_{ij}\geq c$. That is H is transitive. Hence the proof

On Fuzzy M-Normed Matrices:

We shall consider F fuzzy algebra [0, 1] with operations (+,*) and standard order $\leq$ where a+b=max {a, b}, a.b=min {a, b} for all a, b in F. F is a commutative semi ring with additive and multiplicative identies 0 and 1, respectively. Let $M_{mm}(F)$ denotes the set of all $m\times n$ fuzzy matrices over F. In short $M_n(F)$ is the set of all fuzzy matrices of order n. Define '+' and scalar multiplication in $M_n(F)$ as $A+B=[a_{ij}+b_{ij}]$ where $A=[a_{ij}]$ and $B=[b_{ij}]$ and $CA=[Ca_{ij}]$ where $C\in[0,1]$ with these operations $M_n(F)$ forms a vector space over [0, 1]. In all vector space more properties can be analysed if the vecter spaces anr supplied with matrices. The matrices are defined in vecter spaces through the introduction of suitable non negative quantity called norm. In $M_n(F)$ also the same technique is adoted by introducing the concept norm in the following way.

Fuzzy M-Normed and Semimetric:

Let $M_n(F)$ be the set of all $(n\times n)$ fuzzy matrices over F =[0,1].

For every A in $M_n(F)$ define m-norm of A denoted by $\|A\|_m$ as $\|A\|_m = \max[a_{ij}]$ where $A=[a_{ij}]$ (or) $=\max[a_{11}, a_{12}, ...a_{ij}, ...a_{mn}]$ (Or) $=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$.

Theorem:

If $M_n(F)$ is the set of all $(n \times n)$ fuzzy matrices over F=[0,1] then for all fuzzy matrices A and B in $M_n(F)$ and any scalar C in [0,1] We have,

1. $\|A\|_m > 0$ *and* $\|A\|_m = 0$ *iff if* $A = 0$
2. $\|CA\|_m = C\|A\|_m$ for any C in [0, 1].
3. $\|A+B\|_m = \|A\|_m + \|B\|_m$ for A, B in $M_n(F)$.
4. $\|AB\|_m = \|A\|_m \|B\|_m$ for A, B in $M_n(F)$.

Proof:

Let A=$[a_{ij}]$ and B=$[b_{ij}]$ be two fuzzy matrices.

1. Since all $a_{ij} \in [0,1]$ max$[a_{ij}]$= $\|A\|_m \geq 0$ for all $A \in M_n(F) \Rightarrow a_{ij} = 0$ For all i and j$\Rightarrow$A=0 Contradiction if A=0 then max. $[a_{ij}] = 0 \Rightarrow \|A\|_m = 0$. $\therefore$ $\|A\|_m = 0$ Iff if A=0

2. If $C \in [0,1]$ Then CA=$[Ca_{ij}]$,

$$\therefore \begin{aligned} \|CA\|_m &= \max[Ca_{ij}] = C\max[a_{ij}] \\ &= C\|A\|_m \end{aligned}$$

3. $\|A\|_m$ =max $[a_{ij}]$ and $\|B\|_m$ =max $[a_{ij}]$. Now $\|A+B\|_m$ =max$[C_{ij}]$ where $[C_{ij}]$=$[a_{ij}]$
=max $\{[a_{ij}]+[b_{ij}]\}$ =max$[a_{ij}]$+max$[b_{ij}]$ =$\|A\|_m + \|B\|_m$.

4. $\|A\|_m$ =max$[a_{ij}] = a_{ij}$ and $\|B\|_m$ =max $[b_{ij}] = b_{ij}$ If AB=D then the entries of D are given by, $d_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n}\{\min(a_{ik}, b_{kj})\}$=min $((a_{i1}, b_{1j})$ +min (a_{i2}, b_{2j}) +.........+min (a_{in}, b_{nj}) (1)

Case (i): If all $a_{ij} \leq [b_{ij}]$ for j=1, 2....n Then we have $d_{ij} = a_{i1} + a_{i2} + ... + a_{in} = a_{ij}$

$$\therefore \max[d_{ij}] = \max[a_{ij}]$$

(i.e.) $\|AB\|_m = \|A\|_m = \|A\|_m \|B\|_m$

Case (ii): If all $b_{ij} \leq a_{ij}$, for j=1,2,....n Then we have $d_{ij} = b_{i1} + b_{i2} + ... + b_{in}$ $(from(1)) = b_{ij}$

$$\therefore \max[d_{ij}] = \max[b_{ij}]$$

(i.e.) $\|AB\|_m = \|B\|_m = \|A\|_m \|B\|_m$

Case (iii): Let some $a_{ij} \leq [b_{ij}]$ and some other $b_{ij} \leq a_{ij}$ without loss of generality. Let we assume that $a_{im} < b_{im}$ for all n<m and $b_{im} < a_{im}$ for all $n \geq m$

$$\therefore \text{From (1) } d_{ij} = a_{ij} + ...a_{im} + b_{i(m+1)} + ... + b_{in}$$

$$d_{ij} = \sum_{j=1}^{m} a_{ij} + \sum_{j=m+1}^{n} b_{ij} = a_{ij} + b_{ij}$$

$$\begin{aligned} d_{ij} &= a_{ij},\ if\ \ a_{ij} \geq b_{ij} \\ &= b_{ij},\ if\ \ a_{ij} \leq b_{ij} \end{aligned}$$

$\therefore$ Max $[[d_{ij}]$ =max$[a_{ij}]$=$\|A\|_m$ (Or) Max$[b_{ij}]$=$\|B\|_m$

$$\|AB\|_m = \|A\|_m \|B\|_m.$$

Theorem:

The above mapping d satisfies the following conditions for all A, B, C in M_n (F).

1. d(A,B) ≥ 0, if d(A,B)=0 then A=B
2. d(A,B)=d(B,A)
3. d(A,B) $\leq$ d(A,C)+d(B,C) for all A,B,C

M_n (F) Thus d is a pseudo-metric in M_n (F).

Proof:

1. d(A,B)= $\| A+B \|_m \geq 0 \quad \forall$ A,B in M_n (F), $\therefore$ d(A,B) ≥ 0

Suppose d (A, B) =0 then $\| A+B \|_m = 0 \Rightarrow \quad \| A \|_m + \| B \|_m = 0 \Rightarrow$ A=0 and B=0 $\Rightarrow$ A=B

But A=B $\Rightarrow$ $\| A \|_m = \| B \|_m$ (i.e.) $\| A \|_m + \| B \|_m = \| B \|_m + \| B \|_m = \| B \|_m \Rightarrow \| A+B \|_m = \| B \|_m$

$\Rightarrow$ d (A,B) $\neq 0$

$\therefore$ A=B need not imply d (A, B) =0

2. d(A,B)= $\| A+B \|_m = \| B+A \|_m$ = d(B,A), $\therefore$ d (A,B)=d(B,A)

3. Let A,B,C in M_n (F) be such that $\| C \|_m \geq \| B \|_m \geq \| A \|_m$

$$d(A,B) = \| A+B \|_m = \| A \|_m + \| B \|_m = \| B \|_m$$
$$d(A,C) = \| A+C \|_m = \| A \|_m + \| C \|_m = \| C \|_m$$
$$d(B,C) = \| B+C \|_m = \| B \|_m + \| C \|_m = \| C \|_m$$
$$d(A,B) = \| B \| \leq \| C \|_m = d(A,C)+d(B,C), \therefore d(A,B) \leq d(A,C)+d(B,C)$$

Similarly for the other cases also we have d(A,B) $\leq$ d(A,C)+d(B,C). Thus in all cases, d(A,B) $\leq$ d(B,C)+d(C,A) for all A,B,C in M_n (F). Thus from (i) (ii) and (iii) we see that d is a pseudo-metric on M_n (F). The above pseudo=metric can be extended to a finite product of M_n (F).

Theorem:

The mapping d: X×X→[0,1] defined as $d(\overline{A},\overline{B}) = \sum_{I=1}^{n} d_i(A_i, B_i)$, where $\overline{A} = A_1, A_2, \ldots\ldots A_n$ and $\overline{B}$ = ($B_1, B_2, \ldots\ldots B_n$) are in V and d_i are pseudo-metric on M_n (F) is a pseudo-metric for X.

Proof:

Since d_i's are pseudo-matrices. $d_i(A_i, B_i) \geq 0$, i=1, 2,.....n

$\therefore \quad \sum d_i(A_i, B_i) \geq 0$ (i.e.) $d(\overline{A},\overline{B}) \geq 0 \quad \forall \; \overline{A}, \overline{B} \in X$

If $d(\overline{A},\overline{B})$ =0 then $\sum_{i=1}^{n} d_i(A_i, B_i) = 0 \quad \Rightarrow$ Max $d_i(A_i, B_i) = 0 \Rightarrow d_i(A_i, B_i) = 0$, for i=1, 2,....n $\Rightarrow \| A_i + B_i \|_m = 0 \Rightarrow \| A_i \|_m + \| B_i \|_m = 0$

A_i = 0 and B_i = 0 for i=1,2,...n, $\therefore (A_1, A_2, \ldots\ldots A_n) = (B_1, B_2, \ldots\ldots B_n)$

(i.e.) $\overline{A} = \overline{B}$. Thus $d(\overline{A},\overline{B})$ =0 $\Rightarrow \overline{A} = \overline{B}$

Conversely if $\overline{A} = \overline{B}$ then $A_i = B_i$, for i=1,2 ,...n

$A_i = B_i \Rightarrow \| A_i \|_m = \| B_i \|_m$ for i=1, 2,.....n

(i.e.) $\| A_i \|_m + \| B_i \|_m = \| B_i \|_m + \| B_i \|_m$ for i=1,2,.....n

(i.e.) $\| A_i + B_i \|_m = \| B_i \|_m$ for i=1, 2,......n

(i.e.) $d_i(A_i, B_i) = \| B_i \|_m \geq 0$

$\Rightarrow \sum_{I=1}^{n} d_i(A_i, B_i)$ Need not be equal to zero (i.e.) $d(\overline{A}, \overline{B})$ need not be zero

$$\therefore \quad \overline{A} = \overline{B} \neq d(\overline{A}, \overline{B}) = 0.$$

$$d(\overline{A}, \overline{B}) = \sum_{i=1}^{n} d_i(\overline{A_i}, \overline{B_i}) = \sum_{I=1}^{n} d_i'(\overline{B_i}, \overline{A_i}) = d(\overline{B}, \overline{A})$$

For each d_i, i=1,2,.....n $d_i(A_i, B_i) \leq d_i(A_i, B_i) + d_i(C_i, B_i)$

$$\sum_{i=1}^{n} d_i(A_i, B_i) \leq \sum_{i=1}^{n} d_i(A_i, C_i) + \sum_{i=1}^{n} d_i(C_i, B_i) \text{ (i.e.) } d(\overline{A}, \overline{B}) < d(\overline{A}, \overline{C}) + d(\overline{C}, \overline{B})$$

Thus from (i),(ii), (iii) and (iv) we see that d is pseudo-metric. For X= M_n (F) $\times$,.............. M_n (F) (n times).

Theorem:

If A, A', B, B' in M_n (F). Then d(A, B)+d(A', B')=d(A, A')+d(B, B').

Proof:

$$\text{d(A, b)+d(A', B')} = \| A+B \|_m + \| A'+B' \|_m$$

$$\| A \|_m + \| B \|_m + \| A' \|_m + \| B' \|_m + \| A+A' \|_m \| B+B' \|_m = \text{d(A, A')+d(B, B')}.$$

Hence the proof

Matrices Generated Form the Semi-Metric on Fuzzy Matrices:

Theorem:

The relation' $\sim$ ' is an equivalence relation on $M_n(F)$.

Proof:

$$\text{d(A,A)=0} \Leftrightarrow \| A+A \|_m = 0 \Leftrightarrow \| A \|_m + \| A \|_m = 0 \Leftrightarrow \| A \|_m = 0 \Leftrightarrow \text{A=0}$$

A $\sim$ A $\forall$ A in $M_n(F)$ $\therefore$ Reflexivity is true,

A $\sim$ B d(A,B)=0 $\Rightarrow$ d(B,A)=0 $\Rightarrow$ B $\sim$ A

(i.e.) A $\sim$ B $\Rightarrow$ B $\sim$ A $\therefore$ Symmetric is true.

A $\sim$ B $\Rightarrow$ d(A,B)=0 $\Rightarrow$ $\| A \|_m = \| Bi \|_m$

B $\sim$ C $\Rightarrow$ d (B, C) =0 $\Rightarrow$ $\| Bi \|_m = \| C \|_m$

C $\sim$ A $\Rightarrow$ d(C, A) =0 $\Rightarrow$ $\| C \|_m = \| A \|_m$

$\therefore$ A $\sim$ B and B $\sim$ C $\Rightarrow$ A $\sim$ C $\therefore$ Transitive is true. $\therefore$ $\sim$ is an equivalence relation on $M_n(F)$.

Theorem:

The mapping f: C $\times$ C $\rightarrow$ [0, 1] defined as f {[A], [B]} =d (A, B) is a metric on C.

Proof:

To show that f is well- defined (i.e.) To show that f is independent of elements chosen. If $A' \in$ [A] and $B' \in$ [B] then A' $\sim$ A and B' $\sim$ B (i.e.) d (A, A') =0 and d (B, B') =0 $\Rightarrow$ d(A, B) + $d(A', B')$=0 $\Rightarrow$ d(A, B) = $d(A', B')$=0

(i.e.) f {[A], [B]} =f {[A'],[B']} Thus f is well defined.

f{[A],[B]} $\geq$ 0 since d(A,B) $\geq$ 0, f{[A],[B]}=0, iff [A]=[B]. For [A] = [B] $\Rightarrow$ A $\sim$ B $\Rightarrow$ d (A, B) =0, (i.e.) f {[A], [B]} = 0

Conversely, f {[A], [B]} = 0, $\Rightarrow$ D (A,B) = 0 $\Rightarrow$ A $\sim$ B $\Rightarrow$ [A] = [B]

f{[A],[B]} = d(A,B) $\leq$ d(A,C)+d(C,B) $\leq$ f{[A],[C]}+f{[C],[B]}

(i.e.) f {[A], [B]} $\leq$ f{[A],[C]}+f{[C],[B]} for every [A],[B],[C] $\in$ C

Thus f is a metric on C. From (ii) (iii) and (iv) f is a natural metric on C.

Thus we get a full metric space C form a pseudo-metric space $M_n(F)$. (i.e.) (c, f) is a metric space.

On- Regularity of Block Triangular Fuzzy Matrix:

Let F[0, 1] be fuzzy algebra over the support [0, 1] with operations + and - defined as a+b= max{a,b} and a-b= min{a,b} for all $a,b \in [0,1]$ and then standard order $\geq$. Let F_{mn} be the set of all $m \times n$ fuzzy matrices over F. A matrix $A \in F_{mn}$ is set to be regular if there exists $X \in F_{mn}$ such that AXA =A . In this case X is called a generalized inverse, each element $a \in F$ is regular, because axa =a holds under the fuzzy multiplication for all $x \geq a$. Hence F is regular, it is well known that for arbitrary ring R, R is regular. However the study on regularity of fuzzy matrices. Algorithms for a fuzzy matrix to be regular is given, finite fuzzy relational equation can be expressed in the norm of fuzzy matrix equations as x.A=b, for some co-efficients matrix $A \in F_{mn}$ and $b \in F_{In}$. In the solution of a fuzzy matrix equation whose co-efficient matrix is regular has been discussed. If A is regular with the generalized inverse X then b. X is solution of x.A=b. Further every invertible matrix is regular. Regular fuzzy matrix plays an important role in estimation and inverse problem in fuzzy relation equation and fuzzy optimization problems. In fuzzy retrieval system the degree of relevance of the concept matrix depends on that of its transitive closure which is a regular matrix.

Lemma:

For $A, B \in F_{mn}$ the following statements hold (i) $R(B) \subseteq R(A) \Leftrightarrow$ There exists $X \in F_{mn}$ such that B=XA. (ii) $C(B) \subseteq C(A) \Leftrightarrow$ There exists $Y \in F_{mn}$ such that B=AY.

Regular Block Triangular Fuzzy Matrices:

In this section we derive equivalent conditions for regularity of block triangular fuzzy matrix of the form,

$$M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \text{ With } R(C) \subseteq R(A) \text{ and } C(C) \subseteq C(D) \ldots\ldots\ldots (1)$$

First we prove certain lemmas that simplify the proofs of the main results. "I" denote the identity matrix of appropriate size.

Lemma:

For $A, B \in F_{mn}$ if A is regular then (i) R (B) $\subseteq$ R(A) $\Leftrightarrow$ B = B, $A^{-}A$ for each A^{-} of A. (ii) C (B) $\subseteq$`C (A) $\Leftrightarrow$ B = A, $A^{-}B$ for each A^{-} of A.

Proof:

(i) According to lemma R (B)$\subseteq$R (A) $\Leftrightarrow$ there exist $X \in F_{mm}$ such that B =XA. By definition A=A A^{-}A. Hence B=XA$\Rightarrow$B =X.A A^{-}A=B A^{-}A conversely, Suppose B=B A^{-}A then we have B =XA by taking X=B A^{-}. Hence R (B)$\subseteq$R(A) $\Leftrightarrow$B=B A^{-}A. Can be proved in the same manner.

Lemma:

Let M=$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ be a lower block triangular matrix. The following statements hold.

(i) If R(C) $\subseteq$ R(A) then $\rho_r(M) = \rho_r(A) + \rho_r(D)$

(ii) If C(C) $\subseteq$C(D) then $\rho_c(M) = \rho_c(A) + \rho_c(D)$.

Proof:

From lemma it follows that if R(C) $\subseteq$ R(A) Then there exist X such that C=XA we express M in the form,

$$M=\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}=UL$$

Where, $U=\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$ and $L=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$

$$\rho_r(M)=\rho_r(UL)\le\rho_r(L)=\rho_r(A)+\rho_r(D)$$

$$\text{Since } M=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ we get } \rho_r(M) > \rho_r(A)+\rho_r(D)$$

Hence $\rho_r(M)=\rho_r(A)+\rho_r(D)$. Can be proved in the same manner.

Theorem:

For any block triangular matrix M of the form M is a rectangular matrix M has a lower block triangular –inverse iff the blocks A and D are regular matrices.

Furthermore $\rho(M)=\rho(A)+\rho(D)$ and $M^-=\begin{pmatrix} A^- & 0 \\ D^-CA^- & D^- \end{pmatrix}$……… (2)

Proof:

Since M^- is a lower block triangular g-inverse of M of the form (2) and M^-MM^- is a semi inverse of M equality the corresponds blocks in $M^-MM^-=M^-$ we get $A^-AA^-=A^-$ and $D^-DD^-=D^-$. Hence A^- and D^- are inverse of A and D respectively. Then $M^==\begin{pmatrix} A^= & 0 \\ D^=CA^= & D^= \end{pmatrix}$ is a semi inverse of M. It is a clear that if M is regular, then it has a g-inverse and a semi inverse. For any fuzzy matrix if the Moore Penrose exists. Then by theorem it is unique and coincides with its transpose. Hence a lower block triangular fuzzy matrix cannot have a lower block triangular Moore Penrose inverse. Hence A and D are regular matrices.

Theorem:

Let M be of the form $M=\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, A and D be square matrices. Then the group inverse $M^{\neq}$ exists iff the group inverse $A^{\neq}$ and $D^{\neq}$ exists and DC=CA.

Proof:

Since M has the form $M=\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, it follows from theorem, and corollary that,

$$MM^-=\begin{pmatrix} AA^= & 0 \\ CA^= & DD^= \end{pmatrix} \text{ and } M^=M=\begin{pmatrix} A^=A & 0 \\ D^=C & D^=D \end{pmatrix}$$

According to the definition of group inverse in $M^{\neq}$ exists, then $MM^==M^=M$. That is $AA^==A^=A$, $D^=D=DD^=$ and $CA^==D^=C$. It means that $A^{\neq}$ and $D^{\neq}$ exist and DC = D ($CA^=$ A) = D ($CA^=$)A = D ($D^=C$ A) = ($DD^=C$) A = CA . The converse follows by retracting the steps.

Fuzzy Relational Equations:

In this section we discuss consistency of fuzzy matrix equation x.M=b. Where $M=\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ is a lower block triangular fuzzy matrix, and x=[y z] and b=[c d] are partitions of x and b respectively in conformity that of M.

Theorem:

For the matrix $M=\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ such that $R(C)\subseteq R(A)$ the blocks A and D are regular matrices then the following statements are equivalent (i) x.M=b is solvable (ii) y.A=c, z.D=d are solvable and $c\ge dD^-C$.

Proof:

(i) $\Rightarrow$(ii) Suppose x.M=b is solvable. Let $\alpha = [\beta\gamma]$ be a solution. The sub suit gives $\beta.A + \gamma.C = c$ and $\gamma.D = d$ Since R(C) $\subseteq$R (A), by using c=$CA^{=}$A we get $(\beta + \gamma CA^{-})A$=c and $\gamma.D = d$. Therefore y.A=c and z.D=d are both solvable with the fact that y=$\beta + \gamma CA^{-}$ is a solution of y.A =c and Z=γ is solution of z.D=d. Since D is regular $\gamma - dD$ is solution of z.D=d. Now $\gamma C = dD^{-}C$ from $\beta A + \gamma C$ =c. By fuzzy addition we get $c \geq \gamma C = dD^{-}C$ as required.

(ii) $\Rightarrow$(i) Suppose y.A=c and z.D=d are solvable. Since both A and D are regular matrices, y= CA^{-} and z=dD^{-} are the solutions of the equation y.A=c and z.D=d respectively. Hence CA^{-}A=c and D^{-}D=d. By fuzzy addition, $c \geq dD^{-}C$ implies $C + dD^{-}C = c$.

$$[cA^{-}\ dD^{-}]\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = [CAA^{-} + dD^{-}CdD^{-}D]$$
$$= [C + dD^{-}C.d]$$
$$= [c\ d] = b$$

Thus $[cA^{-}dD^{-}]$ is a solution of the equation x.M=b. Hence x.M=b is solvable.

Triangular Toeplitz Fuzzy Matrix:

In this section, first we desire equivalent conditions for the ideompotent of a triangular Toeplitz fuzzy matrix of orders upto 3 then we discuss the idempotency of a general triangular toeplitz fuzzy pf order K of the form,

$$\begin{bmatrix} a & 0 & 0 & . & . & . & . & 0 \\ a_1 & a & 0 & . & . & . & . & 0 \\ a_2 & a_1 & a & . & . & . & . & 0 \\ a_{k-2} & a_{k-3} & a_{k-4} & . & . & .. & . & a.0 \\ a_{k-1} & a_{k-2} & a_{k-3} & . & . & . & . & a_1.a \end{bmatrix}.$$

Theorem:

1. Each $a \in [0,1]$ is idempotent as well as regular.
2. $T_2 = \begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix}$ is an idempotent matrix $\Leftrightarrow$ $a_1 \leq a$
3. $\begin{pmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{pmatrix}$ is an idempotent matrix $\Leftrightarrow$ $a_1 \leq a_2 \leq a$.

Proof:

1. Trivial

2. T_2 is an idempotent matrix $\Leftrightarrow$ $\begin{pmatrix} a & 0 \\ a_1a & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix}$ $\Leftrightarrow$ $a_1a = a_1$ $\Leftrightarrow$ $a_1 \leq a$

3. Let us position $T_3 = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ Where, A=$\begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix}$, D= [a], C=$[a_2\ a_1]$.

Conclusion:

In this see we give some basic matrix theory essential to make the book a self contained one. However the book of Paul. Horst on matrix algebra for social scientists would be a boon to social scientists who wish to make use of matrix theory in their analysis. We give some very basic matrix algebra. This is need for the development of fuzzy matrix theory and the psychological problems. However these fuzzy models have been used by applied mathematicians, to study social and psychological problems. These models are very much used by doctors, engineers, scientists, industrialists and statisticians. Here we proceed on to give some basic properties of matrix theory.

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