International Journal of Current Research and Modern Education (IJCRME)
ISSN (Online): 2455 - 5428
(www.rdmodernresearch.com) Volume I, Issue II, 2016

CUBIC IDEALS OF Γ- SEMIGROUPS

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Abstract:

In this paper, we defined new notion of cubic ideals of Γ-semigroups, which is generalized concept of fuzzy ideals of Γ-semigroups. We also investigated some of its properties with examples.

Index Terms - Semigroup, Γ-Semigroup, Regular Γ-Semigroup, Ideal, Bi-Ideal, Interior Ideal, Cubic Ideal, Cubic Bi-Ideal & Cubic Interior Ideal

1. Introduction:

Zadeh [16] initiated the concept of fuzzy sets in 1965. In 1975, Zadeh [17] made an extension concept of a fuzzy set by an interval-valued fuzzy set. A semigroup is an algebraic structure consisting of a non-empty sets together with an associative binary operation. The formal study of semigroups began in the early 20th century. In 1981 Sen [14] introduced the notion of Γ–semigroup as a generalization of semigroup and ternary semigroup. Many results of semigroups could be extended to Γ–semigroups directly and via operator semigroups of a Γ–semigroups. Many authors have studied semigroups in terms of fuzzy sets. Kuroki [10] is the main contributor of this study. Motivated by Kuroki [10] Sardar et al. [13] have initiated the study of Γ–semigroups in terms of fuzzy sets. Kuroki [10] introduced the notion of fuzzy ideals and fuzzy bi-ideals in semigroups. Atanassov [1] introduced intuitionistic fuzzy set is characterized by a membership function and a non-membership function for each element in the Universe. In 2010, K. Hur and H.W. Kang [4] introduced interval-valued fuzzy subgroups and rings. Jun et al. [7] introduced the new concept called cubic sets. These structures encompass interval-valued fuzzy set and fuzzy set. Also Jun et al. [6] introduced the notion of cubic subgroups. Vijayabalaji et al. [15] introduced the notion of cubic linear space. V. Chinnadurai et al. [3] introduced cubic ring. The purpose of this paper to introduce the notion of cubic ideals of Γ–semigroups and we provide some results on it.

2. Introduction:

Now we recall some known concepts related to cubic ideals of Γ- semigroups from the literature, which will be needed in the sequel.

Definition 2.1: [12] Let S be a semigroup. By a subsemigroup of S, we mean a non-empty subset A of S such that $A^2 \subseteq A$.

Definition 2.2: [12] A non-empty subset A of a Γ- semigroup S is said to be a Γ-subsemigroup of S if $A\Gamma A \subseteq A$.

Definition 2.3: [2] A non-empty subset A of a Γ- semigroup S is called left (right) ideal of S such that $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$). If A is both a left and a right ideal of a Γ- semigroup S, then we say A is an ideal of S.

Definition 2.4: [2] A Γ-subsemigroup A of a Γ- semigroup S is called a bi-ideal of S if $A\Gamma S\Gamma A \subseteq A$.

Definition 2.5: [2] A Γ-subsemigroup A of a Γ- semigroup S is called an interior ideal of S if $S\Gamma A\Gamma S \subseteq A$.

Definition 2.6: [12] Let S be a Γ- semigroup and S_1 be a Γ_1- semigroup. A pair of mappings f_1:S→S_1 and f_2:Γ→ Γ_1 is said to be a homomorphism from (S, Γ) to (S_1, Γ_1), if $f_1(x\gamma y) = f_1(x)\, f_2(\gamma)\, f_1(y) \quad \forall\, x, y \in S \text{ and } \forall\, \gamma \in \Gamma$.

Definition 2.7: [2] Let X be a non-empty set. A mapping $\bar{\mu}: X \rightarrow D[0,1]$ is called interval-valued fuzzy set, where D[0,1] denote the family of all closed sub intervals of [0, 1] and a mapping $\lambda: X \rightarrow [0,1]$ is a fuzzy set in X.

Definition 2.8: [2] A fuzzy subset μ of X is called a fuzzy left (right) ideal of X, if $\mu(xy) \geq \mu(y)$, $\big(\mu(xy) \geq \mu(x)\big)$ $\forall x, y \in X$.

if μ is both a fuzzy left and a fuzzy right ideal of X, then μ is called a fuzzy ideal of X.

Definition 2.9: [2] An interval-valued fuzzy subset $\bar{\mu}$ of X is called a interval-valued fuzzy left (right) ideal of X, if $\bar{\mu}(xy) \geq \bar{\mu}(y)$, $\big(\bar{\mu}(xy) \geq \bar{\mu}(x)\big), \forall x, y \in X$. If $\bar{\mu}$ is both an i-v fuzzy left and i-v fuzzy right ideal of X, then $\bar{\mu}$ is called an i-v fuzzy ideal of X.

Definition 2.10: [2] A fuzzy subset μ of X is called a fuzzy bi-ideal of X, if
$i)$ $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$
$ii)$ $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}, \forall x, y, z \in X$

Definition 2.11: [2] A fuzzy subset μ of X is called a fuzzy interior ideal of X, if
$i)$ $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$
$ii)$ $\mu(xyz) \geq \mu(y)\ \forall x, y, z \in X$.

Definition 2.12: [2] A fuzzy subset μ of a Γ- semigroup S is called a fuzzy Γ-subsemigroup of S, if $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$. $\forall\, x, y \in S$ and $\forall\, \gamma \in \Gamma$.

Definition 2.13: [7] Let X be a non empty set. A cubic set $\mathcal{A}$ in X is a structure of the form $\mathcal{A}$=$\{\langle x, \bar{\mu}_A(x), \lambda(x)\rangle : x \in X\}$ and denoted by $\mathcal{A}$=$\langle \bar{\mu}_A, \lambda\rangle$, where $\bar{\mu}_A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set (briefly, IVF) in X and λ is a fuzzy set in X.

Definition 2.14: [7] The complement of $\mathcal{A}$= $\langle \bar{\mu}_A, \lambda\rangle$ is defined by $\mathcal{A}^C = \{\langle x, (\bar{\mu}_A)^C(x), 1-\lambda(x)\rangle \mid x \in X\}$.

Definition 2.15: [7] For any $\mathcal{A}_i = \{\langle x, \bar{\mu}_i(x), \lambda_i(x)\rangle | x \in X\}$ where i ∈⋀ (index set), we have the following,

$$i)\ \cap_{R,i\in\wedge} \mathcal{A}_i = \{\langle x, (\cap_{i\in\wedge} \bar{\mu}_i)(x), (\cup_{i\in\wedge} \lambda_i)(x)\rangle | x \in X\} \quad (R-intersection)$$

$$ii)\ \cup_{R,i\in\wedge} \mathcal{A}_i = \{\langle x, (\cup_{i\in\wedge} \bar{\mu}_i)(x), (\cap_{i\in\wedge} \lambda_i)(x)\rangle | x \in X\} \quad (R-union)$$

$$iii)\ \cap_{P,i\in\wedge} \mathcal{A}_i = \{\langle x, (\cap_{i\in\wedge} \bar{\mu}_i)(x), (\cap_{i\in\wedge} \lambda_i)(x)\rangle | x \in X\} \quad (P-intersection)$$

$$iv)\ \cup_{P,i\in\wedge} \mathcal{A}_i = \{\langle x, (\cup_{i\in\wedge} \bar{\mu}_i)(x), (\cup_{i\in\wedge} \lambda_i)(x)\rangle | x \in X\} \quad (P-union)$$

Definition 2.16: [13] Let $\mathcal{A} = <\bar{\mu}, \lambda>$ is a cubic set in X, then $\mathcal{A} = <\bar{\mu}, \lambda>$ is a cubic KU-ideal of X if and only if for all $\tilde{t} \in D[0,1]$ and $s \in [0,1]$, the set $U(\mathcal{A}\,; \tilde{t}, s)$ is either empty or a KU-ideal of X.

Definition 2.17: [8] For any non-empty subset G of a set X, the characteristic cubic set of G is defined to be a structure $\chi_G(x) =< x, \bar{\mu}_{\chi_G}(x), \gamma_{\chi_G}(x): x \in X >$ which is briefly denoted by $\chi_G(x) =< \bar{\mu}_{\chi_G}(x), \gamma_{\chi_G}(x) >$.

$$\text{Where } \bar{\mu}_{\chi_G}(x) = \begin{cases}[1,1] & if\ x \in G\\ [0,0] & otherwise\end{cases} \text{ and } \gamma_{\chi_G}(x) = \begin{cases}0 & if\ x \in G\\ 1 & otherwise\end{cases}$$

3. Main Results:

In this section, we introduced the notion of cubic ideals of Γ-semigroups and discuss some of its properties. Throughout this paper S stands for Γ- semigroup unless otherwise specified.

Definition 3.1: A non-empty cubic subset $\mathcal{A} = <\bar{\mu}, \omega>$ of S is called a cubic left (right) ideal of S, if it satisfies
$i)$ $\bar{\mu}(x\gamma y) \geq \bar{\mu}(y), (\bar{\mu}(x\gamma y) \geq \bar{\mu}(x))$
$ii)$ $\omega(x\gamma y) \leq \omega(y), (\omega(x\gamma y) \leq \omega(x))$ $\forall\, x, y \in S$ and $\forall\, \gamma \in \Gamma$.

A non-empty cubic subset $\mathcal{A} = < \bar{\mu}, \omega >$ of S is called a cubic ideal of S, if it is a cubic left ideal and a cubic right ideal of S.

Definition 3.2: A non-empty cubic subset $\mathcal{A} =< \bar{\mu}, \omega >$ of S is called a cubic Γ-subsemigroup of S, if it satisfies

$i)\ \bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$,

$ii)\ \omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\}, \quad \forall\, x, y \in S \text{ and } \forall\, \gamma \in \Gamma$.

Definition 3.3: A cubic Γ-subsemigroup $\mathcal{A} = < \bar{\mu}, \omega >$ of S is called a cubic bi-ideal of S, if it satisfies

$i)\ \bar{\mu}(x\alpha y\beta z) \geq \min\{\bar{\mu}(x), \bar{\mu}(z)\}$

$ii)\ \omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(z)\}\ \forall\, x, y, z \in S \text{ and } \forall\, \alpha, \beta \in \Gamma$.

Example 3.4: Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be the non-empty set of binary operations defined below:

γ	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	c
c	0	0	0	b

α	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	0	0	0	c

β	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

Clearly S is a Γ- semigroup. Moreover,

Define an interval-valued fuzzy set $\bar{\mu}$:S→D[0,1] by,

$\bar{\mu}$(0)=[0.8,0.9], $\bar{\mu}$(a)=[0.5,0.6], $\bar{\mu}$(b)=[0.3,0.4] and $\bar{\mu}$(0)=[0.1,0.2] is an interval-valued fuzzy bi-ideal of S.

Define a fuzzy set ω:S→[0,1] by, ω(0)=0.2, ω(a)=0.4, ω(b)=0.6 and ω(c)=0.8 is a fuzzy bi-ideal of S.

Hence $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic bi-ideal of Γ-semigroup S.

Definition 3.5: A cubic Γ-subsemigroup $\mathcal{A} = < \bar{\mu}, \omega >$ of S is called a cubic interior ideal of S, if it satisfies

$i)\ \bar{\mu}(x\alpha y\beta z) \geq \bar{\mu}(y)$,

$ii)\ \omega(x\alpha y\beta z) \leq \omega(y), \qquad \forall\, x, y, z \in S \text{ and } \forall\, \alpha, \beta \in \Gamma$.

Example 3.6: Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be the non-empty set of binary operations defined below:

γ	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	c
c	0	0	0	b

α	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
C	0	0	0	c

β	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

Clearly S is a Γ- semigroup. Moreover,

Define an interval-valued fuzzy set $\bar{\mu}$:S→D[0,1] by,

$\bar{\mu}$(0)=[0.9,1], $\bar{\mu}$(a)=[0.7,0.8], $\bar{\mu}$(b)=[0.5,0.6] and $\bar{\mu}$(0)=[0.3,0.4] is an interval-valued fuzzy interior ideal of S.

And define an fuzzy set ω:S→[0,1] by, ω(0)=0, ω(a)=0.5, ω(b)=0.8 and ω(c)=1is a fuzzy interior ideal of S.

Hence $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic interior ideal of Γ- semigroup S.

Definition 3.7: *Let* $\mathcal{A}_1 =< \bar{\mu}_1, \gamma_1 >$ and $\mathcal{A}_2 =< \bar{\mu}_2, \gamma_2 >$ be any two cubic sets of S then the following cubic sets of S are defined as follows,

$$(\mathcal{A}_1 \circledcirc \mathcal{A}_2)(x) = \begin{cases} (\bar{\mu}_1 \tilde{\circ} \bar{\mu}_2)(x) = \begin{cases} \sup_{x \le p\gamma q} \min\{\bar{\mu}(p), \bar{\mu}(q)\} & \text{for all } p, q \in S, \gamma \in \Gamma \\ [0,0] & \text{otherwise} \end{cases} \\ (\omega_1 \circ \omega_2)(x) = \begin{cases} \inf_{x \le p\gamma q} \max\{\omega(p), \omega(q)\} & \text{for all } p, q \in S, \gamma \in \Gamma \\ 1 & \text{otherwise} \end{cases} \end{cases}$$

$$(\mathcal{A}_1 \circledast \mathcal{A}_2)(x) = \begin{cases} (\bar{\mu}_1 \tilde{*} \bar{\mu}_2)(x) = \begin{cases} \sup_{x = a\gamma b} \min\{\bar{\mu}_1(a), \bar{\mu}_2(b)\} & \forall\, a, b \in S, \gamma \in \Gamma \\ [0,0] & \text{otherwise} \end{cases} \\ (\gamma_1 * \gamma_2)(x) = \begin{cases} \inf_{x = a\gamma b} \max\{\gamma_1(a), \gamma_2(b)\} & \forall\, a, b \in S, \gamma \in \Gamma \\ 1 & \text{otherwise} \end{cases} \end{cases}$$

Definition 3.8: Let $\mathcal{A} = <\bar{\mu}, \omega>$ be a cubic interior ideal of S. Define $U(\mathcal{A}\,; \tilde{t}, n) = \{x \in S | \bar{\mu}(x) \ge \tilde{t} \text{ and } \omega(x) \le n\}$, where $\tilde{t} \in D[0,1]$ and $n \in [0,1]$ is called the level set of $\mathcal{A}$.

Theorem 3.9: Let $\mathcal{A} = <\bar{\mu}, \omega>$ be a cubic ideal of S. If S is an intra regular, then $\mathcal{A}(\mathrm{a}) = \mathcal{A}(\mathrm{a\beta a})$ for all $\mathrm{a} \in \mathrm{S}, \beta \in \Gamma$.

Proof: Let a be any element of S. Since S is intra regular then there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

Such that $a = x\alpha a\beta a\gamma y$, then $\bar{\mu}(a) = \bar{\mu}(x\alpha a\beta a\gamma y) \ge \bar{\mu}(x\alpha a\beta a) \ge \bar{\mu}(a\beta a) \ge \bar{\mu}(a)$ and $\omega(a) = \omega(x\alpha a\beta a\gamma y) \le \omega(x\alpha a\beta a) \le \omega(a\beta a) \le \omega(a)$

Thus $\bar{\mu}(a) = \bar{\mu}(a\beta a)$ and $\omega(a) = \omega(a\beta a)$.

Hence $\mathcal{A}(\mathrm{a}) = \mathcal{A}(\mathrm{a\beta a})$.

Theorem 3.10: Let $\mathcal{A} = <\bar{\mu}, \omega>$ be a cubic ideal of S. If S is an intra regular, then $\mathcal{A}(\mathrm{a\beta b}) = \mathcal{A}(\mathrm{b\beta a})$ for all $\mathrm{a} \in \mathrm{S}, \beta \in \Gamma$.

Proof: Let $a, b \in S$ and $\beta \in \Gamma$. By the above theorem, we have
$\bar{\mu}(a\beta b) = \bar{\mu}(a\beta b\beta a\beta b) \ge \bar{\mu}(a\beta(b\beta a)\beta b) \ge \bar{\mu}(b\beta a)$
$\bar{\mu}(b\beta a) = \bar{\mu}(b\beta a\beta b\beta a) \ge \bar{\mu}(b\beta(a\beta b)\beta a) \ge \bar{\mu}(a\beta b)$ and
$\omega(a\beta b) = \omega(a\beta b\beta a\beta b) \le \omega(a\beta(b\beta a)\beta b) \le \omega(b\beta a)$
$\omega(b\beta a) = \omega(b\beta a\beta b\beta a) \le \omega(b\beta(a\beta b)\beta a) \le \omega(a\beta b)$
Hence $\bar{\mu}(a\beta b) = \bar{\mu}(b\beta a)$ and $\omega(a\beta b) = \omega(b\beta a)$
Therefore $\mathcal{A}(a\beta b) = \mathcal{A}(\mathrm{b\beta a})$.

Proposition 3.11: Let $\mathcal{A}_1 = <\bar{\mu}_1, \omega_1>$ be a cubic right ideal of S and $\mathcal{A}_2 = <\bar{\mu}_2, \omega_2>$ be a cubic left ideal of S, then $\mathcal{A}_1 \circledcirc \mathcal{A}_2 \sqsubseteq \mathcal{A}_1 \sqcap \mathcal{A}_2$.

Proof: Let $x \in S$. Suppose there exist $p, q \in S$ and $\gamma \in \Gamma$, such that $x \le p\gamma q$. Then

$$\begin{aligned}(\bar{\mu}_1 \tilde{\circ} \bar{\mu}_2)(x) &= \sup_{x \le p\gamma q} \min\{\bar{\mu}_1(p), \bar{\mu}_2(q)\} \\ &\le \sup_{x \le p\gamma q} \min\{\bar{\mu}_1(p\gamma q), \bar{\mu}_2(p\gamma q)\} \\ &\le \min\{\bar{\mu}_1(x), \bar{\mu}_2(x)\}\end{aligned}$$

$(\bar{\mu}_1 \tilde{\circ} \bar{\mu}_2)(x) \le (\bar{\mu}_1 \cap \bar{\mu}_2)(x)$, suppose x cannot be expressed as $x \le p\gamma q$, then $(\bar{\mu}_1 \tilde{\circ} \bar{\mu}_2)(x) = \bar{0} \le (\bar{\mu}_1 \cap \bar{\mu}_2)(x)$, this implies that $(\bar{\mu}_1 \tilde{\circ} \bar{\mu}_2)(x) \le (\bar{\mu}_1 \cap \bar{\mu}_2)(x)$ and

$$\begin{aligned}(\omega_1 \circ \omega_2)(x) &= \inf_{x \le p\gamma q} \max\{\omega_1(p), \omega_2(q)\} \\ &\ge \inf_{x \le p\gamma q} \max\{\omega_1(p\gamma q), \omega_2(p\gamma q)\} \\ &\ge \max\{\omega_1(x), \omega_2(x)\}\end{aligned}$$

$(\omega_1 \circ \omega_2)(x) \ge (\omega_1 \cup \omega_2)(x)$, suppose x cannot be expressed as $x \le p\gamma q$, then $(\omega_1 \circ \omega_2)(x) = 1 \ge (\omega_1 \cup \omega_2)(x)$, this implies that $(\omega_1 \circ \omega_2)(x) \ge (\omega_1 \cup \omega_2)(x)$
Hence $\mathcal{A}_1 \circledcirc \mathcal{A}_2 \sqsubseteq \mathcal{A}_1 \sqcap \mathcal{A}_2$.

Proposition 3.12: Let S be a regular Γ-semigroup, let $\mathcal{A}_1 = <\bar{\mu}_1, \omega_1>$ be a cubic right ideal of S and $\mathcal{A}_2 = <\bar{\mu}_2, \omega_2>$ be a cubic left ideal of S, then $\mathcal{A}_1 \circledcirc \mathcal{A}_2 \sqsupseteq \mathcal{A}_1 \sqcap \mathcal{A}_2$.

Proof: Let $x \in S$. Since S is regular Γ-semigroup, then there exist $p \in S$ *and* $\alpha, \beta \in \Gamma$, such that $x = x\alpha p\beta x = x\gamma x$, where $\gamma = \alpha p\beta \in \Gamma$. Then

$(\bar{\mu}_1 \tilde{\circ} \bar{\mu}_2)(x) = \sup_{x=p\gamma q} \min\{\bar{\mu}_1(p), \bar{\mu}_2(q)\}$

$\geq \min\{\bar{\mu}_1(x), \bar{\mu}_2(x)\}$

$(\bar{\mu}_1 \tilde{\circ} \bar{\mu}_2)(x) \geq (\bar{\mu}_1 \cap \bar{\mu}_2)(x)$ and

$(\omega_1 \circ \omega_2)(x) = \inf_{x=p\gamma q} \max\{\omega_1(p), \omega_2(q)\}$

$\leq max\{\omega_1(x), \omega_2(x)\}$

$(\omega_1 \circ \omega_2)(x) \leq (\omega_1 \cup \omega_2)(x)$

Hence $\mathcal{A}_1 \circledcirc \mathcal{A}_2 \sqsupseteq \mathcal{A}_1 \sqcap \mathcal{A}_2$.

Lemma 3.13: If $\mathcal{A} = <\bar{\mu}, \omega>$ is a cubic subset of S, then the following are equivalent:

i) $\bar{\mu} \tilde{*} \bar{\mu} \leq \bar{\mu}$ *and* $\omega * \omega \geq \omega$

ii) $\bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ *and* $\omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\}, \forall x, y \in S$ *and* $\gamma \in \Gamma$.

Proof: Let x, y $\in S$ *and* $\gamma \in \Gamma$. *i)* $\rightarrow$ *ii)*

Consider $(\bar{\mu} \tilde{*} \bar{\mu})(x\gamma y) = \sup_{x\gamma y = a\gamma_1 b} \min\{\bar{\mu}(a), \bar{\mu}(b)\} \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$

By (i), $\bar{\mu} \tilde{*} \bar{\mu} \leq \bar{\mu}$

$\bar{\mu}(x\gamma y) \geq (\bar{\mu} * \bar{\mu})(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$.

Hence $\bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and

$(\omega * \omega)(x\gamma y) = \inf_{x\gamma y = a\gamma_1 b} \max\{\omega(a), \omega(b)\} \leq \max\{\omega(x), \omega(y)\}$

By (i), $\omega * \omega \geq \omega$

$\omega(x\gamma y) \leq (\omega * \omega)(x\gamma y) \leq \max\{\omega(x), \omega(y)\}$

Hence $\omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\}$.

ii) $\rightarrow$ *i)* Consider $(\bar{\mu} \tilde{*} \bar{\mu})(x) = \sup_{x=a\gamma_1 b} \min\{\bar{\mu}(a), \bar{\mu}(b)\} \leq \sup_{x=a\gamma_1 b} \bar{\mu}(a\gamma_1 b) \leq \bar{\mu}(x)$

This implies that $\bar{\mu} \tilde{*} \bar{\mu} \leq \bar{\mu}$

If x cannot be expressed as $x = a\gamma_1 b$, then $(\bar{\mu} \tilde{*} \bar{\mu})(x) = [0,0] \leq \bar{\mu}(x)$ this implies that $(\bar{\mu} \tilde{*} \bar{\mu})(x) \leq \bar{\mu}(x)$.

Hence $\bar{\mu} \tilde{*} \bar{\mu} \leq \bar{\mu}$.

and $(\omega * \omega)(x) = \inf_{x=a\gamma_1 b} \max\{\omega(a), \omega(b)\} \geq \inf_{x=a\gamma_1 b} \omega(a\gamma_1 b) \geq \omega(x)$

Thus $\omega * \omega \geq \omega$

If x cannot be expressed as $x = a\gamma_1 b$, then $(\omega * \omega)(x) = 1 \geq \omega(x)$ this implies that $(\omega * \omega)(x) \geq \omega(x)$.

Hence $\omega * \omega \geq \omega$.

Therefore $\bar{\mu} \tilde{*} \bar{\mu} \leq \bar{\mu}$ and $\omega * \omega \geq \omega$.

Lemma 3.14: A cubic subset $\mathcal{A} = <\bar{\mu}, \omega>$ of S is a cubic left (right) ideal of S if and only if *i)* $\bar{\mu} \tilde{*} \bar{\mu} \leq \bar{\mu}$ and $\omega * \omega \geq \omega$

ii) $\bar{\mu}_{\chi_H} \tilde{*} \bar{\mu} \leq \bar{\mu}$ and $\omega_{\chi_H} * \omega \geq \omega$, where the characteristic cubic set of S is denoted by $\chi_H = <\bar{\mu}_{\chi_H}, \omega_{\chi_H}>$ of H in S and H is a non-empty subset of S.

Proof: Let $\mathcal{A} = <\bar{\mu}, \omega>$ be a cubic left ideal of S. i) follows by above lemma 3.13.

Let $x \in S$. Suppose $x = a\gamma b$ for $x, a, b \in S$ and $\gamma \in \Gamma$.

$(\bar{\mu}_{\chi_H} \tilde{*} \bar{\mu})(x) = \sup_{x=a\gamma b} \min\{\bar{\mu}_{\chi_H}(a), \bar{\mu}(b)\}$

$= \sup_{x=a\gamma b} \min\{[1,1], \bar{\mu}(b)\}$

$= \sup_{x=a\gamma b} \bar{\mu}(b)$

$\leq \sup_{x=a\gamma b} \bar{\mu}(a\gamma b)$

$\leq \bar{\mu}(x)$

This implies that $(\bar{\mu}_{\chi_H} \tilde{*} \bar{\mu})(x) \leq \bar{\mu}(x)$

If x is not expressible as $x = a\gamma b$, then $(\bar{\mu}_{\chi_H} \tilde{*} \bar{\mu})(x) = [0,0] \leq \bar{\mu}(x)$. Thus $\bar{\mu}_{\chi_H} \tilde{*} \bar{\mu} \leq \bar{\mu}$ and

$\left(\omega_{\chi_H} * \omega\right)(x) = \inf_{x=a\gamma b} \max\{\omega_{\chi_H}(a), \omega(b)\}$
$= \inf_{x=a\gamma b} \max\{0, \omega(b)\}$
$= \inf_{x=a\gamma b} \omega(b)$
$\geq \inf_{x=a\gamma b} \omega(a\gamma b)$
$\geq \omega(x)$

This implies that $\left(\omega_{\chi_H} * \omega\right)(x) \geq \omega(x)$

If x is not expressible as $x = a\gamma b$, then $\left(\omega_{\chi_H} * \omega\right)(x) = 1 \geq \omega(x)$. Thus $\omega_{\chi_H} * \omega \geq \omega$.

Conversely, let us assume that (i) and (ii) holds, by (ii)

Let $x, y \in S$ and $\gamma \in \Gamma$. Then we have,

$\bar{\mu}(x\gamma y) \geq \left(\bar{\mu}_{\chi_H} \tilde{*} \bar{\mu}\right)(x\gamma y)$
$= \sup_{x\gamma y = p\gamma_1 q} \min\{\bar{\mu}_{\chi_H}(p), \bar{\mu}(q)\}$
$\geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}(y)\}$
$= \min\{[1,1], \bar{\mu}(y)\}$

$\bar{\mu}(x\gamma y) \geq \bar{\mu}(y)$

and

$\omega(x\gamma y) \leq \left(\omega_{\chi_H} * \omega\right)(x\gamma y)$
$= \inf_{x\gamma y = p\gamma_1 q} \max\{\omega_{\chi_H}(p), \omega(q)\}$
$\leq \max\{\omega_{\chi_H}(x), \omega(y)\}$
$= \max\{0, \omega(y)\}$

$\omega(x\gamma y) \leq \omega(y)$

Hence $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic left (right) ideal of S.

Theorem 3.15: The intersection of any family of cubic bi-ideals of Γ-semigroup S is a cubic bi-ideal of Γ-semigroup S.

Proof: Let $\{\mathcal{A}_i\}_{i\in\wedge}$ be the family of cubic bi-ideals of S, let $x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

Let $\bar{\mu}(x) = \cap\, \bar{\mu}_i(x) = (\inf \bar{\mu}_i)(x) = \inf \bar{\mu}_i(x)$ and
$\omega(x) = \cup\, \omega_i(x) = (\sup \omega_i)(x) = \sup \omega_i(x)$

$\bar{\mu}(x\gamma y) = \inf \bar{\mu}_i(x\gamma y) \geq \inf\{\min\{\bar{\mu}_i(x), \bar{\mu}_i(y)\}\}$
$= \min\{\inf \bar{\mu}_i(x), \inf \bar{\mu}_i(y)\} = \min\{\bar{\mu}(x), \bar{\mu}(y)\}$

$\omega(x\gamma y) = \sup \omega_i(x\gamma y) \leq \sup\{\max\{\omega_i(x), \omega_i(y)\}\}$
$= \max\{\sup \omega_i(x), \sup \omega_i(y)\} = \max\{\omega(x), \omega(y)\}$

Thus $\cap_{i\in\wedge} \mathcal{A}_i$ is a cubic Γ-subsemigroup of S.

Again $\bar{\mu}(x\alpha y\beta z) = \cap\, \bar{\mu}_i(x\alpha y\beta z) = \inf \bar{\mu}_i(x\alpha y\beta z) \geq \inf \min\{\bar{\mu}_i(x), \bar{\mu}_i(z)\}$
$= \min\{\inf \bar{\mu}_i(x), \inf \bar{\mu}_i(z)\}$
$= \min\{\cap\, \bar{\mu}_i(x), \cap\, \bar{\mu}_i(z)\}$

$\bar{\mu}(x\alpha y\beta z) \geq \min\{\bar{\mu}(x), \bar{\mu}(z)\}$ and

$\omega(x\alpha y\beta z) = \cup\, \omega_i(x\alpha y\beta z) = \sup \omega_i(x\alpha y\beta z) \leq \sup \max\{\omega_i(x), \omega_i(z)\}$
$= \max\{\sup \omega_i(x), \sup \omega_i(z)\}$
$= \max\{\cup\, \omega_i(x), \cup\, \omega_i(z)\}$

$\omega(x\alpha y\beta z) \leq \max\{\omega_i(x), \omega_i(z)\}$

Hence $\cap_{i\in\wedge} \mathcal{A}_i$ is a cubic bi-ideal of S.

Theorem 3.16: A cubic set $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic bi-ideal of S if and only if the level set $U(\mathcal{A}; \tilde{t}, n) = \{x \in S \mid \bar{\mu}(x) \geq \tilde{t} \text{ and } \omega(x) \leq n\}$ is a bi-ideal of S, when it is non-empty.

Proof: Let $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic bi-ideal of S. Then

$\bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and $\omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\}$

Let $x, y \in U(\mathcal{A}; \tilde{t}, n)$ and $\gamma \in \Gamma$, then $\bar{\mu}(x) \geq \tilde{t}, \bar{\mu}(y) \geq \tilde{t}$ and $\omega(x) \leq n, \omega(y) \leq n$.

$\bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\} \geq \min\{\tilde{t}, \tilde{t}\} = \tilde{t}$

$\omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\} \leq \max\{n, n\} = n$
Thus $(x\gamma y) \in U(\mathcal{A}\,; \tilde{t}, n)$
Also, $\bar{\mu}(x\alpha y\beta z) \geq \bar{\mu}(y)$ and $\omega(x\alpha y\beta z) \leq \omega(y)$ Let $x, y, z \in U(\mathcal{A}\,; \tilde{t}, n)$ *and* $\alpha, \beta \in \Gamma$, then $\bar{\mu}(x) \geq \tilde{t}, \bar{\mu}(y) \geq \tilde{t}, \bar{\mu}(z) \geq \tilde{t}$ and $\omega(x) \leq n, \omega(y) \leq n, \omega(z) \leq n$.
$\bar{\mu}(x\alpha y\beta z) \geq \min\{\bar{\mu}(x), \bar{\mu}(z)\} \geq \min\{\tilde{t}, \tilde{t}\} = \tilde{t}$ and
$\omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(z)\} \leq \max\{n, n\} = n$
Thus $(x\alpha y\beta z) \in U(\mathcal{A}\,; \tilde{t}, n)$
Hence $U(\mathcal{A}\,; \tilde{t}, n)$ is a bi-ideal of S.
Conversely, suppose that $U(\mathcal{A}\,; \tilde{t}, n)$ is a bi-ideal of S.
Define $\tilde{t} = \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and $n = \max\{\omega(x), \omega(y)\}$
Let $x, y \in U(\mathcal{A}\,; \tilde{t}, n)$ *and* $\gamma \in \Gamma$ then $x\gamma y \in U(\mathcal{A}\,; \tilde{t}, n)$
$\bar{\mu}(x\gamma y) \geq \tilde{t} \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and $\omega(x\gamma y) \leq n \leq \max\{\omega(x), \omega(y)\}$
Thus $\mathcal{A} = <\bar{\mu}, \omega>$ is a cubic Γ-subsemigroup of S.
Also, define $\tilde{t} = \min\{\bar{\mu}(x), \bar{\mu}(z)\}$ and $n = \max\{\omega(x), \omega(z)\}$
Let $x, y, z \in U(\mathcal{A}\,; \tilde{t}, n)$ *and* $\alpha, \beta \in \Gamma$, then $x\alpha y\beta z \in U(\mathcal{A}\,; \tilde{t}, n)$
$\bar{\mu}(x\alpha y\beta z) \geq \tilde{t} \geq \min\{\bar{\mu}(x), \bar{\mu}(z)\}$ and $\omega(x\alpha y\beta z) \leq n \leq \max\{\omega(x), \omega(z)\}$
Hence $\mathcal{A} = <\bar{\mu}, \omega>$ is a cubic bi-ideal of S.

Theorem 3.17: Let H be a non-empty subset of S. Then H is a bi-ideal of S if and only if the characteristic cubic set $\chi_H = <\bar{\mu}_{\chi_H}, \omega_{\chi_H}>$ of H in S is a cubic bi-ideal of S.

Proof: Let H be a bi-ideal of S. Let $x, y \in S$ and $\gamma \in \Gamma$. Suppose that
$\bar{\mu}_{\chi_H}(x\gamma y) < \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\}$ and
$\omega_{\chi_H}(x\gamma y) > \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\}$ for some $x, y \in S$, and $\gamma \in \Gamma$.
Then $\bar{\mu}_{\chi_H}(x\gamma y) = \bar{0}$, $\bar{\mu}_{\chi_H}(x) = \bar{1} = \bar{\mu}_{\chi_H}(y)$ and $\omega_{\chi_H}(x\gamma y) = 1, \omega_{\chi_H}(x) = 0 = \omega_{\chi_H}(y)$.
This implies that $x, y \in H$. Since H is a Γ-subsemigroup of S then $x\gamma y \in H$.
Thus $\bar{\mu}_{\chi_H}(x\gamma y) = \bar{1}$ and $\omega_{\chi_H}(x\gamma y) = 0$, a contradiction.
Hence $\bar{\mu}_{\chi_H}(x\gamma y) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\}$ and $\omega_{\chi_H}(x\gamma y) \leq \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\}$
Again suppose that $\bar{\mu}_{\chi_H}(x\alpha y\beta z) < \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(z)\}$ and
$\omega_{\chi_H}(x\alpha y\beta z) > \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(z)\}$ for some $x, y, z \in S$ *and* $\alpha, \beta, \gamma \in \Gamma$. Then
$\bar{\mu}_{\chi_H}(x\alpha y\beta z) = \bar{0}$, $\bar{\mu}_{\chi_H}(x) = \bar{1} = \bar{\mu}_{\chi_H}(z)$ and $\omega_{\chi_H}(x\alpha y\beta z) = 1, \omega_{\chi_H}(x) = 0 = \omega_{\chi_H}(z)$.
This implies that $x, z \in H$. Since H is a bi-ideal of S, then $x\alpha y\beta z \in H$.
Thus $\bar{\mu}_{\chi_H}(x\alpha y\beta z) = \bar{1}$ and $\omega_{\chi_H}(x\alpha y\beta z) = 0$, a contradiction.
Hence $\bar{\mu}_{\chi_H}(x\alpha y\beta z) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(z)\}$ and $\omega_{\chi_H}(x\alpha y\beta z) \leq \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(z)\}$
Hence $\chi_H = <\bar{\mu}_{\chi_H}, \omega_{\chi_H}>$ is a cubic bi-ideal of S.
Conversely, assume that $\chi_H = <\bar{\mu}_{\chi_H}, \omega_{\chi_H}>$ is a cubic bi-ideal of S, for any subset H of S.
Let $x, y \in H$ and $\gamma \in \Gamma$, then $\bar{\mu}_{\chi_H}(x) = \bar{1} = \bar{\mu}_{\chi_H}(y)$ and $\omega_{\chi_H}(x) = 0 = \omega_{\chi_H}(y)$.
Since χ_H is a cubic bi-ideal of S, thus $\bar{\mu}_{\chi_H}(x\gamma y) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(y)\} \geq \min\{\bar{1}, \bar{1}\} = \bar{1}$ and $\omega_{\chi_H}(x\gamma y) \leq \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(y)\} \leq \max\{0,0\} = 0$.
This implies that $x\gamma y \in H$, for all $x, y \in S$ and $\gamma \in \Gamma$. Hence H is a Γ-subsemigroup of S.
Let $x, y, z \in H$ and $\alpha, \beta \in \Gamma$, then $\bar{\mu}_{\chi_H}(x) = \bar{1} = \bar{\mu}_{\chi_H}(z)$ and $\omega_{\chi_H}(x) = 0 = \omega_{\chi_H}(z)$.
Since χ_H is a cubic bi-ideal of S, thus
$\bar{\mu}_{\chi_H}(x\alpha y\beta z) \geq \min\{\bar{\mu}_{\chi_H}(x), \bar{\mu}_{\chi_H}(z)\} \geq \min\{\bar{1}, \bar{1}\} = \bar{1}$ and $\omega_{\chi_H}(x\alpha y\beta z) \leq \max\{\omega_{\chi_H}(x), \omega_{\chi_H}(z)\} \leq \max\{0,0\} = 0$.
This implies that $x\alpha y\beta z \in H$, for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.
Hence H is a bi-ideal of S.

Theorem 3.18: Let H be a non-empty subset of S. If $\mathcal{A} = <\bar{\mu}, \omega>$ be a cubic set of S defined by

$$\mathcal{A}(x) = \begin{cases} \bar{\mu}(x) = \begin{cases} [p_1, p_2] & \text{if } x \in H \\ [q_1, q_2] & \text{otherwise} \end{cases} \\ \omega(x) = \begin{cases} 1 - p & \text{if } x \in H \\ 1 - q & \text{otherwise} \end{cases} \end{cases}$$

for all $x \in S, [p_1, p_2], [q_1, q_2] \in D[0,1]$and $p, q \in [0,1]$ with $[p_1, p_2] > [q_1, q_2]$ and $p > q$. Then H is a bi-ideal of S if and only if $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic bi-ideal of S.

Proof: Let $x, y \in H$ and $\gamma \in \Gamma$. From the hypothesis $x\gamma y \in H$.

Assume that H is a bi-ideal of S.

We consider four cases:

i) $x \in H$ *and* $y \in H$

ii) $x \in H$ *and* $y \notin H$

iii) $x \notin H$ *and* $y \in H$

iv) $x \notin H$ *and* $y \notin H$

Case (i) If $x \in H$ *and* $y \in H$ then $\bar{\mu}(x) = [p_1, p_2] = \bar{\mu}(y)$ and $\omega(x) = 1 - p = \omega(y)$.

Since H is a bi-ideal of S. $\bar{\mu}(x\gamma y) = [p_1, p_2] = \min\{[p_1, p_2], [p_1, p_2]\} = \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and $\omega(x\gamma y) = 1 - p = \max\{1 - p, 1 - p\} = \max\{\omega(x), \omega(y)\}$.

Case (ii) If $x \in H$ *and* $y \notin H$. Then $\bar{\mu}(x) = [p_1, p_2], \bar{\mu}(y) = [q_1, q_2]$ and $\omega(x) = 1 - p, \omega(y) = 1 - q$.

Clearly, $\min\{\bar{\mu}(x), \bar{\mu}(y)\} = [q_1, q_2]$ and $\max\{\omega(x), \omega(y)\} = 1 - t$.

Now $\bar{\mu}(x\gamma y) = [p_1, p_2]$ or $[q_1, q_2]$ and $\omega(x\gamma y) = 1 - p$ or $1 - q$ if $x\gamma y \in H$ or $x\gamma y \notin H$. By assumption that $[p_1, p_2] > [q_1, q_2]$ and $p > q$, then $\bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and $\omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\}$.

Similarly we can prove that Case (iii).

Case(iv) If $x \notin H$ *and* $y \notin H$. Then $\bar{\mu}(x) = [q_1, q_2] = \bar{\mu}(y)$ and $\omega(x) = 1 - q = \omega(y)$.

So, $\min\{\bar{\mu}(x), \bar{\mu}(y)\} = [q_1, q_2] = \bar{\mu}(x\gamma y)$ and $\max\{\omega(x), \omega(y)\} = 1 - q = \omega(x\gamma y)$.

Therefore $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic Γ-subsemigroup of S.

Let $x, y, z \in H$ and $\alpha, \beta \in \Gamma$. From the hypothesis $x\alpha y\beta z \in H$.

Case(i) If $x \in H$ *and* $z \in H$ then $\bar{\mu}(x) = [p_1, p_2] = \bar{\mu}(z)$ and $\omega(x) = 1 - p = \omega(z)$.

Since H is a bi-ideal of S. $\bar{\mu}(x\alpha y\beta z) = [p_1, p_2] \geq \min\{[p_1, p_2], [p_1, p_2]\} \geq \min\{\bar{\mu}(x), \bar{\mu}(z)\}$ and $\omega(x\alpha y\beta z) = 1 - p \leq \max\{1 - p, 1 - p\} = \max\{\omega(x), \omega(z)\}$.

Case (ii) If $x \in H$ *and* $z \notin H$. Then $\bar{\mu}(x) = [p_1, p_2], \bar{\mu}(z) = [q_1, q_2]$ and $\omega(x) = 1 - p, \omega(z) = 1 - q$.

Clearly, $\min\{\bar{\mu}(x), \bar{\mu}(z)\} = [q_1, q_2]$ and $\max\{\omega(x), \omega(z)\} = 1 - t$.

Now $\bar{\mu}(x\alpha y\beta z) = [p_1, p_2]$ or $[q_1, q_2]$ and $\omega(x\gamma y) = 1 - p$ or $1 - q$, if $x\alpha y\beta z \in H$ or $x\alpha y\beta z \notin H$. By assumption that $[p_1, p_2] > [q_1, q_2]$ and $p > q$, then $\bar{\mu}(x\alpha y\beta z) \geq \min\{\bar{\mu}(x), \bar{\mu}(z)\}$ and $\omega(x\alpha y\beta z) \leq \max\{\omega(x), \omega(z)\}$.

Similarly we can prove that Case (iii).

Case(iv) If $x \notin H$ *and* $y \notin H$. Then $\bar{\mu}(x) = [q_1, q_2] = \bar{\mu}(z)$ and $\omega(x) = 1 - q = \omega(z)$.

So, $\min\{\bar{\mu}(x), \bar{\mu}(z)\} = [q_1, q_2] = \bar{\mu}(x\alpha y\beta z)$ and $\max\{\omega(x), \omega(z)\} = 1 - q = \omega(x\alpha y\beta z)$.

Therefore $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic bi-ideal of S.

Conversely, assume that $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic bi-ideal of S. Let $x, y \in H$, *and* $\gamma \in \Gamma$. Such that $\bar{\mu}(x) = [p_1, p_2] = \bar{\mu}(y)$ and $\omega(x) = 1 - p = \omega(y)$.

By hypothesis $\bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\} = [p_1, p_2]$ and

$$\omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\} = 1 - p.$$

So, $x\gamma y \in H$. Therefore H is a Γ-subsemigroup of S.

Again, let $x, y, z \in H$, *and* $\alpha, \beta \in \Gamma$.

Such that $\bar{\mu}(x) = [p_1, p_2] = \bar{\mu}(z)$ and $\omega(x) = 1 - p = \omega(z)$.

Then $\bar{\mu}(x\alpha y\beta z) = [p_1, p_2] \geq \min\{\bar{\mu}(x), \bar{\mu}(z)\}$ and $\omega(x\alpha y\beta z) = 1 - p \leq \max\{\omega(x), \omega(z)\}$.

So, $x\alpha y\beta z \in H$. Therefore H is a bi-ideal of S.

Theorem 3.19: If $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic bi-ideal of S, then $\mathcal{A}^c = < (\bar{\mu})^c, (\omega)^c >$ is also a cubic bi-ideal of S.

Proof: Let $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic bi-ideal of S and let $x, y \in S, \gamma \in \Gamma$, then

$(\bar{\mu})^c(x\gamma y) = 1 - \bar{\mu}(x\gamma y) \leq 1 - \min\{\bar{\mu}(x), \bar{\mu}(y)\} = \max\{1 - \bar{\mu}(x), 1 - \bar{\mu}(y)\}$

$(\bar{\mu})^c(x\gamma y) \leq \max\{(\bar{\mu})^c(x), (\bar{\mu})^c(y)\}$

$(\omega)^c(x\gamma y) = 1 - \omega(xy) \geq 1 - \max\{\omega(x), \omega(y)\} = \min\{1 - \omega(x), 1 - \omega(y)\}$,

$(\omega)^c(x\gamma y) \geq \min\{(\omega)^c(x), (\omega)^c(y)\}$

Thus $\mathcal{A}^c = < (\bar{\mu})^c, (\omega)^c >$ is a cubic Γ-subsemigroup of S, and

Let $x, y, z \in S,\ and\ \alpha, \beta \in \Gamma$, then

$(\bar{\mu})^c(x\alpha y\beta z) = 1 - \bar{\mu}(x\alpha y\beta z) \leq 1 - \min\{\bar{\mu}(x), \bar{\mu}(z)\} = \max\{1 - \bar{\mu}(x), 1 - \bar{\mu}(z)\}$

$(\bar{\mu})^c(x\alpha y\beta z) \leq \max\{(\bar{\mu})^c(x), (\bar{\mu})^c(z)\}$

$(\omega)^c(x\alpha y\beta z) = 1 - \omega(x\alpha y\beta z) \geq 1 - \max\{\omega(x), \omega(z)\} = \min\{1 - \omega(x), 1 - \omega(z)\}$,

$(\omega)^c(x\gamma y) \geq \min\{(\omega)^c(x), (\omega)^c(y)\}$

Hence $\mathcal{A}^c = < (\bar{\mu})^c, (\omega)^c >$ is a cubic bi-ideal of S.

Proposition 3.20: Let $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic ideal of S, then $\mathcal{A}$ is a cubic interior ideal of S.

Proof: Let $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic ideal of S, $x, y \in S\ and\ \gamma \in \Gamma$. Then,

$\bar{\mu}(x\gamma y) \geq \bar{\mu}(x)$ and $\bar{\mu}(x\gamma y) \geq \bar{\mu}(y)$ which implies that

$\bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$.

Again, we have

$\omega(x\gamma y) \leq \omega(x)$ and $\omega(x\gamma y) \leq \omega(y)$ which implies that

$\omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\}$.

Hence $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic Γ-subsemigroup of S.

Now let $x, y, z \in S\ and\ \alpha, \beta \in \Gamma$. Then,

$\bar{\mu}(x\alpha y\beta z) = \bar{\mu}(x\alpha(y\beta z)) \geq \bar{\mu}(y\beta z) \geq \bar{\mu}(y)\ and$

$\omega(x\alpha y\beta z) = \omega(x\alpha(y\beta z)) \leq \omega(y\beta z) \leq \omega(y)$

Thus $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic interior ideal of S.

Proposition 3.21: Let S be a regular Γ- semigroup and $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic interior ideal of S, then $\mathcal{A}$ is a cubic ideal of S.

Proof: Let $x, y \in S\ and\ \gamma \in \Gamma$. Since S is regular, for any $x \in S$ there exist $a \in S$, such that $x = x\alpha a\beta x$.

Then, $\bar{\mu}(x\gamma y) = \bar{\mu}(x\alpha a\beta x\gamma y) \geq \bar{\mu}(x)$ and

$\omega(x\gamma y) = \omega(x\alpha a\beta x\gamma y) \leq \omega(x)$

So, $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic right ideal of S.

Similarly, we can prove that $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic left ideal of S.

Hence $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic ideal of S.

Remark 3.22: From the above two propositions it is clear that in regular Γ-semigroups the concept of cubic ideals and cubic interior ideals are coincide.

Theorem 3.23: If $\{\mathcal{A}_i\}_{i\in\wedge}$ is a family of cubic interior ideals of S, then $\cap_{i\in\wedge} \mathcal{A}_i$ is a cubic interior ideal of S.

Proof: Let $\{\mathcal{A}_i\}_{i\in\wedge}$ be the family of cubic interior ideals of S.

Let $x, y, z \in S\ and\ \alpha, \beta, \gamma \in \Gamma$ and

let $\bar{\mu}(x) = \cap\, \bar{\mu}_i(x) = (inf\ \bar{\mu}_i)(x) = \inf \bar{\mu}_i(x)$

$\omega(x) = \cup\, \omega_i(x) = (sup\ \omega_i)(x) = \sup \omega_i(x)$,

$\bar{\mu}(x\gamma y) = \inf \bar{\mu}_i(x\gamma y) \geq \inf\{\min\{\bar{\mu}_i(x), \bar{\mu}_i(y)\}\}$

$= \min\{\inf \bar{\mu}_i(x), \inf \bar{\mu}_i(y)\} = \min\{\bar{\mu}(x), \bar{\mu}(y)\}$

$\omega(x\gamma y) = \sup \omega_i(x\gamma y) \leq \sup\{\max\{\omega_i(x), \omega_i(y)\}\}$

$= \max\{\sup \omega_i(x), \sup \omega_i(y)\} = \max\{\omega(x), \omega(y)\}$

Thus $\cap_{i\in\wedge} \mathcal{A}_i$ is a cubic Γ-subsemigroup of S.

Again, $\bar{\mu}(y) = \inf \bar{\mu}_i(y) \leq \inf\{\bar{\mu}_i(x\alpha y\beta z)\} = \bar{\mu}(x\alpha y\beta z)$ and

$\omega(y) = \sup \omega_i(y) \geq \sup\{\omega_i(x\alpha y\beta z)\} = \omega(x\alpha y\beta z)$

Hence $\cap_{i\in\wedge} \mathcal{A}_i$ is a cubic interior ideals of S.

Theorem 3.24: Let S be a Γ- semigroup then $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic interior ideal of S if and only if the level set $U(\mathcal{A}\,; \tilde{t}, n) = \{x \in S | \bar{\mu}(x) \geq \tilde{t} \text{ and } \omega(x) \leq n\}$ is an interior ideal of S, when it is non-empty.

Proof: Let $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic interior ideal of S. Then

$i)\ \bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$,

$ii)\ \omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\}$

Let $x, y \in U(\mathcal{A}\,; \tilde{t}, n)$ and $\gamma \in \Gamma$, then $\bar{\mu}(x) \geq \tilde{t}, \bar{\mu}(y) \geq \tilde{t}$ and $\omega(x) \leq n, \omega(y) \leq n$.

$$\bar{\mu}(x\gamma y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\} \geq \min\{\tilde{t}, \tilde{t}\} = \tilde{t}$$

$$\omega(x\gamma y) \leq \max\{\omega(x), \omega(y)\} \leq \max\{n, n\} = n$$

Thus $(x\gamma y) \in U(\mathcal{A}\,; \tilde{t}, n)$

Also, $i)\ \bar{\mu}(x\alpha y\beta z) \geq \bar{\mu}(y)$,

$ii)\ \omega(x\alpha y\beta z) \leq \omega(y)$,

Let $x, y, z \in U(\mathcal{A}\,; \tilde{t}, n)$ and $\alpha, \beta \in \Gamma$, then

$\bar{\mu}(x) \geq \tilde{t}, \bar{\mu}(y) \geq \tilde{t}, \bar{\mu}(z) \geq \tilde{t}$ and $\omega(x) \leq n, \omega(y) \leq n, \omega(z) \leq n$.

$\bar{\mu}(x\alpha y\beta z) \geq \bar{\mu}(y) \geq \tilde{t}$ and $\omega(x\alpha y\beta z) \leq \omega(y) \leq n$

Thus $(x\alpha y\beta z) \in U(\mathcal{A}\,; \tilde{t}, n)$

Hence $U(\mathcal{A}\,; \tilde{t}, n)$ is an interior ideal of S.

Conversely, suppose that $U(\mathcal{A}\,; \tilde{t}, n)$ is an interior ideal of S.

Define $\tilde{t} = \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and $n = \max\{\omega(x), \omega(y)\}$.

Let $x, y \in U(\mathcal{A}\,; \tilde{t}, n)$ and $\gamma \in \Gamma$ then $x\gamma y \in U(\mathcal{A}\,; \tilde{t}, n)$

$\bar{\mu}(x\gamma y) \geq \tilde{t} \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}$ and $\omega(x\gamma y) \leq n \leq \max\{\omega(x), \omega(y)\}$

Thus $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic Γ-subsemigroup of S.

Also, define $\tilde{t} = \bar{\mu}(y)$ and $n = \omega(y)$

Let $x, y, z \in U(\mathcal{A}\,; \tilde{t}, n)$ and $\alpha, \beta \in \Gamma$, then $x\alpha y\beta z \in U(\mathcal{A}\,; \tilde{t}, n)$

$\bar{\mu}(x\alpha y\beta z) \geq \tilde{t} \geq \bar{\mu}(y)$ and $\omega(x\alpha y\beta z) \leq n \leq \omega(y)$

Hence $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic interior ideal of S.

4. Homomorphism of Cubic Ideals of Γ- Semigroups:

Definition 4.1: [2] A mapping f from a Γ- semigroup P to another Γ- semigroup Q is called a homomorphism, if $f(x\gamma y) = f(x)\gamma f(y) \quad \forall\, x, y \in S \text{ and } \forall\, \gamma \in \Gamma$.

Definition 4.2: [7] Let P and Q be given classical sets. A mapping f: P→Q induces two mappings $\mathrm{C}_f: \mathrm{C}(P) \to \mathrm{C}(Q), \mathcal{A}_1 \to \mathrm{C}_f(\mathcal{A}_1)$ and $\mathrm{C}_f^{-1}: \mathrm{C}(Q) \to \mathrm{C}(P), \mathcal{A}_2 \to \mathrm{C}_f^{-1}(\mathcal{A}_2)$.

Where the mapping C_f is called cubic transformation and C_f^{-1} is called inverse cubic transformation.

Definition 4.3: Let f be a mapping from a set P to Q and $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic set of P then the image of P $\mathrm{C}_f(\mathcal{A}) = < \mathrm{C}_f(\bar{\mu}), \mathrm{C}_f(\omega) >$ is a cubic set of Q is defined by

$$\mathrm{C}_f(\mathcal{A})(x') = \begin{cases} \mathrm{C}_f(\bar{\mu})(x') = \begin{cases} \sup\limits_{f(x)=x'} \bar{\mu}(x) & if\ x' \in Q \\ [0,0] & otherwise \end{cases} \\ \mathrm{C}_f(\omega)(x') = \begin{cases} \inf\limits_{f(x)=x'} \omega(x) & if\ x' \in Q \\ 1 & otherwise \end{cases} \end{cases}$$

and

Let f be a mapping from a set P to Q and $\mathcal{A} = < \bar{\mu}, \omega >$ be a cubic set of Q then the pre image of Q $\mathrm{C}_f^{-1}(\mathcal{A}) = < \mathrm{C}_f^{-1}(\bar{\mu}), \mathrm{C}_f^{-1}(\omega) >$ is a cubic set of P is defined by

$$\mathrm{C}_f^{-1}(\mathcal{A})(x) = \begin{cases} \mathrm{C}_f^{-1}(\bar{\mu}(x)) = \bar{\mu}(f(x)) \\ \mathrm{C}_f^{-1}(\omega(x)) = \omega(f(x)) \end{cases}$$

Theorem 4.4: Let f: P→Q be a homomorphism of Γ- semigroup S and let Cf: C(P)→ C(Q) be the cubic transformation induced by f, if $\mathcal{A} = <\bar{\mu}, \omega>$ is a cubic bi- ideal of Q, then $C_f^{-1}(\mathcal{A}) = <C_f^{-1}(\bar{\mu}), C_f^{-1}(\omega)>$ is a cubic bi- ideal of P.

Proof: Let $x, y \in P$ and $\gamma \in \Gamma$. Then,

$C_f^{-1}(\bar{\mu}(x\gamma y)) = \bar{\mu}(f(x\gamma y) = \bar{\mu}(f(x)\gamma f(y))$

(since f is a homomorphism of Γ- semigroups)

$C_f^{-1}(\bar{\mu}(x\gamma y)) \geq \min\{\bar{\mu}(f(x)), \bar{\mu}(f(y))\} = \min\{C_f^{-1}(\bar{\mu})(x), C_f^{-1}(\bar{\mu})(y)\}$

and

$C_f^{-1}(\omega(x\gamma y)) = \omega(f(x\gamma y) = \omega(f(x)\gamma f(y))$

(since f is a homomorphism of Γ- semigroups)

$C_f^{-1}(\omega(x\gamma y)) \leq \max\{\omega(f(x)), \omega(f(y))\} = \max\{C_f^{-1}(\omega)(x), C_f^{-1}(\omega)(y)\}$

Therefore, $C_f^{-1}(\mathcal{A}) = <C_f^{-1}(\bar{\mu}), C_f^{-1}(\omega)>$ is a cubic Γ-subsemigroup of P.

Again, let $x, y, z \in P$ and $\alpha, \beta \in \Gamma$. Then,

$C_f^{-1}(\bar{\mu}(x\alpha y\beta z)) = \bar{\mu}(f(x\alpha y\beta z) = \bar{\mu}(f(x)\alpha f(y)\beta f(z) \geq \min\{\bar{\mu}(f(x), \bar{\mu}(f(z))$

$C_f^{-1}(\bar{\mu}(x\alpha y\beta z)) \geq \min\{C_f^{-1}(\bar{\mu})(x), C_f^{-1}(\bar{\mu})(z)\}$

and

$C_f^{-1}(\omega(x\alpha y\beta z)) = \omega(f(x\alpha y\beta z) = \omega(f(x)\alpha f(y)\beta f(z) \leq \max\{\omega(f(x)), \omega(f(z)\}$

$C_f^{-1}(\omega(x\alpha y\beta z)) \leq \max\{C_f^{-1}(\omega)(x), C_f^{-1}(\omega)(z)\}$

Hence, $C_f^{-1}(\mathcal{A}) = <C_f^{-1}(\bar{\mu}), C_f^{-1}(\omega)>$ is a cubic bi-ideal of P.

Theorem 4.5: Let f: P→Q be a homomorphism of Γ- semigroup S and let Cf: C(P)→ C(Q) be the cubic transformation induced by f,

If $\mathcal{A} = <\bar{\mu}, \omega>$ is a cubic interior ideal of P, then $C_f(\mathcal{A}) = <C_f(\bar{\mu}), C_f(\omega)>$ is a cubic interior ideal of Q.

Proof: Let $\mathcal{A} = <\bar{\mu}, \omega>$ is a cubic interior ideal of P, then it is a cubic Γ-subsemigroup of P.

Let $x', y' \in Q$ and $\gamma \in \Gamma$.

$$C_f(\bar{\mu})(x'\gamma y') = \sup_{f(z)=x'\gamma y'} \bar{\mu}(z)$$
$$\geq \sup_{f(x)=x',\ f(y)=y'} \bar{\mu}(x\gamma y)$$
$$\geq \sup_{f(x)=x',\ f(y)=y'} \min\{\bar{\mu}(x), \bar{\mu}(y)\}$$
$$= \min\{\sup_{f(x)=x'} \bar{\mu}(x), \sup_{f(y)=y'} \bar{\mu}(y)\}$$
$$= \min\{C_f(\bar{\mu})(x'), C_f(\bar{\mu})(y')\}$$

and

$$C_f(\omega)(x'\gamma y') = \inf_{f(z)=x'\gamma y'} \omega(z)$$
$$\leq \inf_{f(x)=x',\ f(y)=y'} \omega(x\gamma y)$$
$$\leq \inf_{f(x)=x',\ f(y)=y'} \max\{\omega(x), \omega(y)\}$$
$$= \max\{\inf_{f(x)=x'} \omega(x), \inf_{f(y)=y'} \omega(y)\}$$
$$= \max\{C_f(\omega)(x'), C_f(\omega)(y')\}$$

Therefore, $C_f(\mathcal{A}) = <C_f(\bar{\mu}), C_f(\omega)>$ is a cubic Γ-subsemigroup of Q.

Again, let $x', y', z' \in Q$ and $\alpha, \beta \in \Gamma$. Then,

$$C_f(\bar{\mu})(x'\alpha y'\beta z') = \sup_{f(z)=x'\alpha y'\beta z'} \bar{\mu}(z)$$
$$\geq \sup_{f(x)=x',\ f(y)=y',\ f(z)=z'} \bar{\mu}(x\alpha y\beta z)$$
$$\geq \sup_{f(y)=y'} \bar{\mu}(y)$$

$C_f(\bar{\mu})(x'\alpha y'\beta z') \geq C_f(\bar{\mu})(y')$

and

$$\mathrm{C}_f(\omega)(x'\alpha y'\beta z') = \inf_{f(z)=x'\alpha y'\beta z'} \omega(z)$$

$$\leq \inf_{f(x)=x',\ f(y)=y', f(z)=z'} \omega(x\alpha y\beta z)$$

$$\leq \inf_{f(y)=y'} \omega(y)$$

$\mathrm{C}_f(\omega)(x'\alpha y'\beta z') \leq \mathrm{C}_f(\omega)(y')$

Hence $\mathrm{C}_f(\mathcal{A}) = < \mathrm{C}_f(\bar{\mu}), \mathrm{C}_f(\omega) >$ is a cubic interior ideal of Q.

Theorem 4.6: Let f: P→Q be a homomorphism of Γ- semigroup S and let C_f: C(P)→ C(Q) be the cubic transformation induced by f,

If $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic interior ideal of Q, then $\mathrm{C}_f^{-1}(\mathcal{A}) = < \mathrm{C}_f^{-1}(\bar{\mu}), \mathrm{C}_f^{-1}(\omega) >$ is a cubic interior ideal of P.

Proof: Let $\mathcal{A} = < \bar{\mu}, \omega >$ is a cubic interior ideal of Q, then it is a cubic Γ-subsemigroup of Q.

Let $x, y \in P$ and $\gamma \in \Gamma$. Then,

$\mathrm{C}_f^{-1}\big(\bar{\mu}(x\gamma y)\big) = \bar{\mu}(f(x\gamma y) = \bar{\mu}(f(x)\gamma f(y))$

$\geq \min\{\bar{\mu}(f(x)), \bar{\mu}(f(y))\}$

$\mathrm{C}_f^{-1}\big(\bar{\mu}(x\gamma y)\big) = \min\{\mathrm{C}_f^{-1}(\bar{\mu})(x), \mathrm{C}_f^{-1}(\bar{\mu})(y)\}$

and

$\mathrm{C}_f^{-1}\big(\omega(x\gamma y)\big) = \omega(f(x\gamma y) = \omega(f(x)\gamma f(y))$

$\leq \max\{\omega(f(x)), \omega(f(y))\}$

$\mathrm{C}_f^{-1}\big(\omega(x\gamma y)\big) = \max\{\mathrm{C}_f^{-1}(\omega)(x), \mathrm{C}_f^{-1}(\omega)(y)\}$

Therefore, $\mathrm{C}_f^{-1}(\mathcal{A}) = < \mathrm{C}_f^{-1}(\bar{\mu}), \mathrm{C}_f^{-1}(\omega) >$ is a cubic Γ-subsemigroup of P.

Again, let $x, y, z \in P$ and $\alpha, \beta \in \Gamma$. Then,

$\mathrm{C}_f^{-1}\big(\bar{\mu}(x\alpha y\beta z)\big) = \bar{\mu}(f(x\alpha y\beta z) = \bar{\mu}(f(x)\alpha f(y)\beta f(z) \geq \bar{\mu}(f(y))$

$\mathrm{C}_f^{-1}\big(\bar{\mu}(x\alpha y\beta z)\big) \geq \mathrm{C}_f^{-1}(\bar{\mu})(y)$

and

$\mathrm{C}_f^{-1}\big(\omega(x\alpha y\beta z)\big) = \omega(f(x\alpha y\beta z) = \omega(f(x)\alpha f(y)\beta f(z) \leq \omega(f(y))$

$\mathrm{C}_f^{-1}\big(\omega(x\alpha y\beta z)\big) \leq \mathrm{C}_f^{-1}(\omega)(y)$

Hence $\mathrm{C}_f^{-1}(\mathcal{A}) = < \mathrm{C}_f^{-1}(\bar{\mu}), \mathrm{C}_f^{-1}(\omega) >$ is a cubic interior ideal of P.

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