International Journal of Current Research and Modern Education (IJCRME)
Impact Factor: 6.725, ISSN (Online): 2455 - 5428
(www.rdmodernresearch.com) Volume 2, Issue 1, 2017

OPERATIONS ON P-ORDER AND R-ORDER OF EXTERNAL CUBIC SOFT MATRICES

V. Chinnadurai* & S. Barkavi**
Department of Mathematics, Annamalai University, Annamalai Nagar, Chidambaram, Tamilnadu

Cite This Article: V. Chinnadurai & S. Barkavi, "Operations on P-Order and R-Order of External Cubic Soft Matrices", International Journal of Current Research and Modern Education, Volume 2, Issue 1, Page Number 132-155, 2017.

Abstract:

In this paper, we introduce the notion of P-order (R-order) of external cubic soft matrices. Further, we develop union and intersection of P-order (R-order) of external cubic soft matrices and their related properties have been investigated.

Key Words: Cubic Soft Sets, Cubic Soft Matrices, P-Order External Cubic Soft Matrices & R-Order External Cubic Soft Matrices.

1. Introduction:

Fuzzy set theory was proposed by Lotifi A. Zadeh [12] and it has extensive applications in various fields. In 1999, Molodstov[8] introduced the novel concept of soft sets and established the fundamental results of the new theory. In 2003, Maji *et al.*[6] studied some properties of soft sets. Pei and Miao [10] and Chen [1]*et al.*, improved the work of Maji *et al.* [5, 6]. In [3], Jun *et al.*, introduced a new notion of cubic set which is a combination of fuzzy set and interval-valued fuzzy set and also investigated several properties of cubic sets.

Muhiuddin and Al-roqi [9], introduced the concepts of internal, external cubic soft sets, P-cubic (R-cubic) soft subsets, R-union(R-intersection, P-union and P-intersection) of cubic soft sets and the complement of a cubic soft sets. They also investigated several related properties and applied the notion of cubic soft sets to BCK/BCI-algebras.

Fuzzy matrix was introduced by Thomason [11] and the concept of uncertainty was discussed by using fuzzy matrices. Different concepts and ideas of fuzzy matrices have been given earlier mainly by Kim, Meenakshi and Thomson [4, 7, 11]. Fuzzy matrix plays a vital role in fuzzy modeling, fuzzy diagnosis and fuzzy controls. It also has applications in fields like psychology, medicine, economics and sociology.

In the early investigation, Chinnadurai and Barkavi introduced a new concepts of cubic soft matrix, internal cubic soft matrix, external cubic soft matrix and discussed various result in[2].

In this paper, the new definition of P-order external cubic soft matrices and R-order external cubic soft matrices have been introduced, along with some algebraic properties are discussed with suitable illustrations.

2. Preliminaries:

In this section we recall some basic definitions and results which will be needed in the sequel.

Definition 2.1 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set over U is defined to be a pair (Φ, A) where Φ is a mapping from A to $P(U)$ and $A \subseteq E.$ Then the pair (Φ, A) can be represented as, $(\Phi, A) = \{\Phi(e)/e \in A\}$ where $\Phi(e) = \{\langle u, \tilde{A}_e(u), \lambda_e(u)\rangle / u \in U, e \in A\}$ is a cubic soft set in which $\tilde{A}_e(u)$ is the interval valued fuzzy set and $\lambda_e(u)$ is a fuzzy set.

Definition 2.2 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set (Φ, A) over U is said to be an internal cubic soft set if $A_e^-(u) \leq \lambda_e(u) \leq A_e^+(u)$ for all $e \in A$ and for all $u \in U$.

Definition 2.3 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set (Φ, A) over U is said to be an external cubic soft set if $\lambda_e(u) \notin (A_e^-(u), A_e^+(u))$ for all $e \in A$ and for all $u \in U$.

Definition 2.4 [9] Let U be an initial universal set and E be a set of parameters. For any subsets A and B of E, (Φ, A) and (Γ, B) be cubic soft sets over U.

1. The R-union of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cup B$ and

$$\text{H}e) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cup_R \Gamma(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$ and it is denoted by $(\mathrm{H}, C) = (\Phi, A) \cup_R (\Gamma, B)$.

2. The R-intersection of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cap B$ and

$$\mathrm{H}e) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cap_R \Gamma(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$ and it is denoted by $(\mathrm{H}, C) = (\Phi, A) \cap_R (\Gamma, B)$.

Definition 2.5 [9] Let U be an initial universal set and E be a set of parameters. For any subsets A and B of E, (Φ, A) and (Γ, B) be cubic soft sets over U.

1. The P-union of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cup B$ and

$$\mathrm{H}e) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cup_P \Gamma(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathrm{H}, C) = (\Phi, A) \cup_P (\Gamma, B)$.

2. The P-intersection of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cap B$ and

$$\mathrm{H}e) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cap_P \Gamma(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathrm{H}, C) = (\Phi, A) \cap_P (\Gamma, B)$.

Definition 2.6 [9] Let U be an initial universal set and E be a set of parameters. The complement of a cubic soft set (Φ, A) over U is denoted by $(\Phi, A)^c$ and defined by

$(\Phi, A) = (\Phi^c, \neg A)$ where $\Phi^c : \neg A \to XP(U)$ and $(\Phi, A)^c = \{\Phi^c(e)/e \in A\}$ where

$\Phi^c(e) = \{\langle u, \tilde{A}_e^c(u), \lambda_e^c(u)\rangle / u \in U, e \in A\}$

Definition 2.7 [7] A matrix $A = [a_{ij}]_{m \times n}$ is said to be fuzzy matrix if $a_{ij} \in [0,1], 1 \le i \le m$ and $1 \le j \le n$.

Definition 2.8 [7] For any two fuzzy matrices $A = [a_{ij}], B = [b_{ij}]$ and a scalar $k \in F$. Then,

(i) $A + B = [sup\{a_{ij}, b_{ij}\}] = \vee[a_{ij}, bij]$.

(ii) $AB = [sup\{inf\{a_{ij}, b_{ij}\}\}] = \vee\{\wedge[a_{ij}, b_{ij}]\}$.

(iii) $kA = [inf\{k, a_{ij}\}] = \wedge[k, a_{ij}]$.

Definition 2.9 [2] Let $U = \{u_1, u_2, \ldots, u_m\}$ be an initial universal set and $E = \{e_1, e_2, \ldots, e_n\}$ be a set of parameters. Let $A \subseteq E$. Then cubic soft set (Φ, A) can be expressed in matrix form as

$$A^{(} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

such that $A^{(} = [a_{ij}] = \left\langle \tilde{A}_{e_j}(u_i), \lambda_{e_j}(u_i) \right\rangle = \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$ which is called an $m \times n$ cubic soft matrix (shortly CS-matrix or CSM) of the cubic soft set (Φ, A).

According to this definition, a cubic soft set (Φ, A) is uniquely characterized by matrix $[a_{ij}]_{m \times n}$ where $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,n$.

Example 2.10 Let $U = \{u_1, u_2, u_3, u_4\}$ is a set of cars and $A = \{e_1, e_2, e_3\}$ is a set of parameters, which stands for mileage, engine and Prize respectively. Then cubic soft set is defined as

$$(\Phi, A) = \{ \; [e_1, (u_1, \langle[0.5,0.8],0.6\rangle), (u_2, \langle[0.1,0.7],0.5\rangle), (u_3, \langle[0.2,0.6],0.9\rangle), (u_4, \langle[0.3,0.9],0.4\rangle)],$$
$$[e_2, (u_1, \langle[0.2,0.5],0.3\rangle), (u_2, \langle[0.3,0.6],0.7\rangle), (u_3, \langle[0.2,0.7],0.2\rangle), (u_4, \langle[0.3,0.5],0.1\rangle)],$$
$$[e_3, (u_1, \langle[0.1,0.8],0.4\rangle), (u_2, \langle[0.6,0.7],0.9\rangle), (u_3, \langle[0.2,0.9],0.5\rangle), (u_4, \langle[0.3,0.7],0.4\rangle)]\}.$$

Then the CS-matrix $\breve{A}$ is written as,

$$\breve{A} = \begin{bmatrix} \langle[0.5,0.8],0.6\rangle & \langle[0.2,0.5],0.3\rangle & \langle[0.1,0.8],0.4\rangle \\ \langle[0.1,0.7],0.5\rangle & \langle[0.3,0.6],0.7\rangle & \langle[0.6,0.7],0.9\rangle \\ \langle[0.2,0.6],0.7\rangle & \langle[0.2,0.7],0.2\rangle & \langle[0.2,0.9],0.5\rangle \\ \langle[0.3,0.9],0.4\rangle & \langle[0.3,0.5],0.1\rangle & \langle[0.3,0.7],0.4\rangle \end{bmatrix}.$$

Definition 2.11 [2] Let $\breve{A} = [a_{ij}] \in CSM_{m\times n}$. Then $\breve{A}$ is an external cubic soft matrix (ECSM), if $\lambda_{ij}^{a} \notin (\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+})$ for all i, j.

Definition 2.12 [2] Let $\breve{A} = [a_{ij}]_{m\times n}$, $\breve{B} = [b_{ij}]_{m\times n}$ be the two cubic soft matrix of order $m \times n$. Then P-order matrix is denoted and defined as $[a_{ij}] \subseteq_P [b_{ij}]$, if $\tilde{A}_{ij}^{a} \le \tilde{B}_{ij}^{b}$ and $\lambda_{ij}^{a} \le \mu_{ij}^{b}$ for all i, j.

Definition 2.13 [2] Let $\breve{A} = [a_{ij}]_{m\times n}$, $\breve{B} = [b_{ij}]_{m\times n}$ be the two cubic soft matrix of order $m \times n$. Then R-order matrix is denoted and defined as $[a_{ij}] \subseteq_R [b_{ij}]$, if $\tilde{A}_{ij}^{a} \le \tilde{B}_{ij}^{b}$ and $\lambda_{ij}^{a} \ge \mu_{ij}^{b}$ for all i, j.

Definition 2.14 [2] Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle \in CSM_{m\times n}$. Then

1. P-union of $\breve{A}$ and $\breve{B}$ is denoted by $\breve{A} \vee_P \breve{B}$ and defined as $\breve{A} \vee_P \breve{B} = \breve{C}$, if $\breve{C} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, where $\tilde{C}_{ij}^{c} = \max\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}$ and $\gamma_{ij}^{c} = \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}$ for all i, j.
2. P-intersection of $\breve{A}$ and $\breve{B}$ is denoted by $\breve{A} \wedge_P \breve{B}$ and defined as $\breve{A} \wedge_P \breve{B} = \breve{C}$, if $\breve{C} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, where $\tilde{C}_{ij}^{c} = \min\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}$ and $\gamma_{ij}^{c} = \min\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}$ for all i, j.
3. R-union of $\breve{A}$ and $\breve{B}$ is denoted by $\breve{A} \vee_R \breve{B}$ and defined as $\breve{A} \vee_R \breve{B} = \breve{C}$, if $\breve{C} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, where $\tilde{C}_{ij}^{c} = \max\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}$ and $\gamma_{ij}^{c} = \min\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}$ for all i, j.
4. R-intersection of $\breve{A}$ and $\breve{B}$ is denoted by $\breve{A} \wedge_R \breve{B}$ and defined as $\breve{A} \wedge_R \breve{B} = \breve{C}$, if $\breve{C} = [c_{ij}] = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$, where $\tilde{C}_{ij}^{c} = \min\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}$ and $\gamma_{ij}^{c} = \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}$ for all i, j.

3. P-order and R-order of External Cubic Soft Matrices.

In this section we define P-order and R-order of external cubic soft matrix and discussed some algebraic properties.

Definition 3.1 Let $\breve{A} = \left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle$ and $\breve{B} = \left\langle \tilde{B}_{ij}^{b}, \mu_{ij}^{b} \right\rangle$ be external cubic soft matrices, if $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^{a} \le \tilde{B}_{ij}^{b}$ and $\lambda_{ij}^{a} \le \mu_{ij}^{b}$ for all i, j. Then it is defined as P-order external cubic soft matrices and it is denoted by $ECSM^{P}$.

Example 3.2 Let $U = \{u_1, u_2\}$ be the universal set and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters. Let $A, B \subseteq E$. Then

$$(\Phi, A) = \{ \; [e_1, (u_1, \langle[0.3,0.5],0.2\rangle), (u_2, \langle[0.4,0.6],0.3\rangle)], \; [e_2, (u_1, \langle[0.1,0.5],0.7\rangle), (u_2, \langle[0.4,0.6],0.2\rangle)]\} \text{ and}$$

$$(\Phi, B) = \{ \; [e_1, (u_1, \langle[0.4,0.6],0.9\rangle), (u_2, \langle[0.7,0.9],1\rangle)], \; [e_2, (u_1, \langle[0.2,0.8],0.85\rangle), (u_2, \langle[0.7,0.9],0.4\rangle)]\}.$$

Then P-order external cubic soft matrices is defined as follows,

$$\breve{A} = \begin{bmatrix} \langle[0.3,0.5],0.2\rangle & \langle[0.1,0.5],0.7\rangle \\ \langle[0.4,0.6],0.3\rangle & \langle[0.4,0.6],0.2\rangle \end{bmatrix} \subseteq_P \breve{B} = \begin{bmatrix} \langle[0.4,0.6],0.9\rangle & \langle[0.2,0.8],0.85\rangle \\ \langle[0.7,0.9],1\rangle & \langle[0.7,0.9],0.4\rangle \end{bmatrix}.$$

(i.e) $\breve{A} \subseteq_P \breve{B}$.

Definition 3.3 Let $\breve{A} = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle$ and $\breve{B} = \langle \tilde{B}^b_{ij}, \mu^b_{ij} \rangle$ be external cubic soft matrices, if $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}^a_{ij} \geq \tilde{B}^b_{ij}$ and $\lambda^a_{ij} \leq \mu^b_{ij}$ for all i, j. Then it is defined as R-order external cubic soft matrices and it is denoted by $ECSM^R$.

Example 3.4 Let $U = \{u_1, u_2\}$ be the universal set and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters. Let $A, B \subseteq E$. Then

$$(\Phi, A) = \{ \ [e_1, (u_1, \langle[0.2,0.4],0.7\rangle), (u_2, \langle[0.4,0.6],0.8\rangle)], \\ [e_2, (u_1, \langle[0.3,0.6],0.2\rangle), (u_2, \langle[0.1,0.4],0.7\rangle)]\}$$

and

$$(\Phi, B) = \{ \ [e_1, (u_1, \langle[0.3,0.6],0.2\rangle), (u_2, \langle[0.6,0.9],0.5\rangle)], \\ [e_2, (u_1, \langle[0.5,0.8],0.1\rangle), (u_2, \langle[0.4,0.7],0.3\rangle)]\}.$$

Then R-order external cubic soft matrices is defined as follows,

$$\breve{A} = \begin{bmatrix} \langle[0.2,0.4],0.7\rangle & \langle[0.3,0.6],0.2\rangle \\ \langle[0.4,0.6],0.8\rangle & \langle[0.1,0.4],0.7\rangle \end{bmatrix} \subseteq_R \breve{B} = \begin{bmatrix} \langle[0.3,0.6],0.2\rangle & \langle[0.5,0.8],0.1\rangle \\ \langle[0.6,0.9],0.5\rangle & \langle[0.4,0.7],0.3\rangle \end{bmatrix}.$$

(i.e) $\breve{A} \subseteq_R \breve{B}$.

Definition 3.5 Let $\breve{A}, \breve{B} \in ECSM^P_{m \times n}$. Then

a) Union of $\breve{A}, \breve{B}$ is defined by $\breve{A} \wedge_P \breve{B} = \breve{C} = [c_{ij}]$,

where $\breve{C} = \langle [\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}], \max\{\lambda^a_{ij}, \mu^b_{ij}\} \rangle$ for all i,j.

b) Intersection of $\breve{A}, \breve{B}$ is defined by $\breve{A} \vee_P \breve{B} = \breve{C} = [c_{ij}]$,

where $\breve{C} = \langle [\min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}], \min\{\lambda^a_{ij}, \mu^b_{ij}\} \rangle$ for all i,j.

Example 3.6 Consider Example 3.2. Then union and intersection of $\breve{A}, \breve{B}$ are given by,

a) $\breve{A} \vee_P \breve{B} = \begin{bmatrix} \langle[0.4,0.6],0.9\rangle & \langle[0.2,0.8],0.85\rangle \\ \langle[0.7,0.9],1\rangle & \langle[0.7,0.9],0.4\rangle \end{bmatrix}.$

b) $\breve{A} \wedge_P \breve{B} = \begin{bmatrix} \langle[0.3,0.5],0.2\rangle & \langle[0.1,0.5],0.7\rangle \\ \langle[0.4,0.6],0.3\rangle & \langle[0.4,0.6],0.2\rangle \end{bmatrix}.$

Definition 3.7 Let $\breve{A}, \breve{B} \in ECSM^R_{m \times n}$. Then

a) Union of $\breve{A}, \breve{B}$ is defined by $\breve{A} \vee_R \breve{B} = \breve{C} = [c_{ij}]$,

where $\breve{C} = \langle [\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}], \min\{\lambda^a_{ij}, \mu^b_{ij}\} \rangle$ for all i,j.

b) Intersection of $\breve{A}, \breve{B}$ is defined by $\breve{A} \wedge_R \breve{B} = \breve{C} = [c_{ij}]$,

where $\breve{C} = \langle [\min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}], \max\{\lambda^a_{ij}, \mu^b_{ij}\} \rangle$ for all i,j.

Example 3.8 Consider Example 3.4. Then union and intersection of $\breve{A}, \breve{B}$ are given by,

a) $\breve{A} \vee_R \breve{B} = \begin{bmatrix} \langle [0.3, 0.6], 0.2 \rangle & \langle [0.5, 0.8], 0.1 \rangle \\ \langle [0.6, 0.9], 0.5 \rangle & \langle [0.4, 0.7], 0.3 \rangle \end{bmatrix}.$

b) $\breve{A} \wedge_R \breve{B} = \begin{bmatrix} \langle [0.2, 0.4], 0.7 \rangle & \langle [0.3, 0.6], 0.2 \rangle \\ \langle [0.4, 0.6], 0.8 \rangle & \langle [0.1, 0.4], 0.7 \rangle \end{bmatrix}.$

Proposition 3.9 *Let* $\breve{A}, \breve{B}, \breve{C} \in ECSM^P_{m \times n}$. *Then*

(i) $\breve{A} \vee_P \breve{A} = \breve{A}$.

(ii) $\breve{A} \vee_P \breve{B} = \breve{B}$.

(iii) $(\breve{A} \vee_P \breve{B}) \vee_P \breve{C} = \breve{A} \vee_P (\breve{B} \vee_P \breve{C}) = \breve{C}$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij} \right\rangle, \breve{B} = \left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij} \right\rangle, \breve{C} = \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij} \right\rangle \in ECSM^P_{m \times n}$.

Then (i) $\breve{A} \vee_P \breve{A} = \left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij} \right\rangle \vee_P \left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij} \right\rangle$ for all i, j.

Since $\breve{A} \subseteq_P \breve{A} \Leftrightarrow \tilde{A}^a_{ij} \leq \tilde{A}^a_{ij}$ and $\lambda^a_{ij} \leq \lambda^a_{ij}$ for all i, j.

By Definition 3.5(a),

$$\begin{aligned} \breve{A} \vee_P \breve{A} &= \left\langle [\max\{\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^-}_{ij}\}, \max\{\tilde{A}^{a^+}_{ij}, \tilde{A}^{a^+}_{ij}\}], \max\{\lambda^a_{ij}, \lambda^a_{ij}\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij} \right\rangle \\ &= \breve{A}. \end{aligned}$$

(ii) $\breve{A} \vee_P \breve{B} = \left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij} \right\rangle \vee_P \left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij} \right\rangle$ for all i, j.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}^a_{ij} \leq \tilde{B}^b_{ij}$ and $\lambda^a_{ij} \leq \mu^b_{ij}$ for all i, j.

By Definition 3.5(a),

$$\begin{aligned} \breve{A} \vee_P \breve{B} &= \left\langle [\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}], \max\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij} \right\rangle \\ &= \left\langle \tilde{B}^b_{ij}, \mu^b_{ij} \right\rangle \\ &= \breve{B}. \end{aligned}$$

(iii) $(\breve{A} \vee_P \breve{B}) \vee_P \breve{C} = \left(\left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij} \right\rangle \vee_P \left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij} \right\rangle \right) \vee_P \breve{C}$.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}^a_{ij} \leq \tilde{B}^b_{ij}$ and $\lambda^a_{ij} \leq \mu^b_{ij}$ for all i, j.

By Definition 3.5(a),

$$\begin{aligned} (\breve{A} \vee_P \breve{B}) \vee_P \breve{C} &= \left(\left\langle [\max\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}, \max\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}], \max\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle \right) \vee_P \breve{C} \\ &= \left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij} \right\rangle \vee_P \breve{C} \\ &= \left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij} \right\rangle \vee_P \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij} \right\rangle. \end{aligned}$$

Since $\breve{B} \subseteq_P \breve{C} \Leftrightarrow \tilde{B}^b_{ij} \leq \tilde{C}^c_{ij}$ and $\mu^b_{ij} \leq \gamma^c_{ij}$ for all i, j.

By Definition 3.5(a),

$$\begin{aligned}\left(\breve{A} \vee_P \breve{B}\right) \vee_P \breve{C} &= \left\langle [\max\{\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}\}, \max\{\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ &= \breve{C}.\end{aligned}$$

Similarly, $\breve{A} \vee_P \left(\breve{B} \vee_P \breve{C}\right) = \breve{C}$.

Hence, $\left(\breve{A} \vee_P \breve{B}\right) \vee_P \breve{C} = \breve{A} \vee_P \left(\breve{B} \vee_P \breve{C}\right) = \breve{C}$.

Proposition 3.10 *Let* $\breve{A}, \breve{B}, \breve{C} \in ECSM_{m\times n}^R$. *Then*

(i) $\breve{A} \vee_R \breve{A} = \breve{A}$.

(ii) $\breve{A} \vee_R \breve{B} = \breve{B}$.

(iii) $\left(\breve{A} \vee_R \breve{B}\right) \vee_R \breve{C} = \breve{A} \vee_R \left(\breve{B} \vee_R \breve{C}\right) = \breve{C}$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \in ECSM_{m\times n}^R$.

Then (i) $\breve{A} \vee_R \breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle$ for all i, j.

Since $\breve{A} \subseteq_R \breve{A} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{A}_{ij}^a$ and $\lambda_{ij}^a \ge \lambda_{ij}^a$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned}\breve{A} \vee_R \breve{A} &= \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{A}_{ij}^{a^+}\}], \min\{\lambda_{ij}^a, \lambda_{ij}^a\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \breve{A}.\end{aligned}$$

(ii) $\breve{A} \vee_R \breve{B} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle$ for all i, j.

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned}\breve{A} \vee_R \breve{B} &= \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \\ &= \breve{B}.\end{aligned}$$

(iii) $\left(\breve{A} \vee_R \breve{B}\right) \vee_R \breve{C} = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right) \vee_R \breve{C}$.

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned}\left(\breve{A} \vee_R \breve{B}\right) \vee_R \breve{C} &= \left(\left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle\right) \vee_R \breve{C} \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \vee_R \breve{C} \\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \vee_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle.\end{aligned}$$

Since $\breve{B} \subseteq_R \breve{C} \Leftrightarrow \tilde{B}_{ij}^b \le \tilde{C}_{ij}^c$ and $\mu_{ij}^b \ge \gamma_{ij}^c$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned}(\breve{A} \vee_R \breve{B}) \vee_R \breve{C} &= \left\langle [\max(\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}), \max(\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+})], \min(\mu_{ij}^b, \gamma_{ij}^c) \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \\ &= \breve{C}.\end{aligned}$$

Similarly, $\breve{A} \vee_R (\breve{B} \vee_R \breve{C}) = \breve{C}$.

Hence, $(\breve{A} \vee_R \breve{B}) \vee_R \breve{C} = \breve{A} \vee_R (\breve{B} \vee_R \breve{C}) = \breve{C}$.

Proposition 3.11 *Let* $\breve{A}, \breve{B}, \breve{C} \in ECSM_{m\times n}^P$. *Then*

(i) $\breve{A} \wedge_P \breve{A} = \breve{A}$.

(ii) $\breve{A} \wedge_P \breve{B} = \breve{A}$.

(iii) $(\breve{A} \wedge_P \breve{B}) \wedge_P \breve{C} = \breve{A} \wedge_P (\breve{B} \wedge_P \breve{C}) = \breve{A}$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \in ECSM_{m\times n}^P$.

Then (i) $\breve{A} \wedge_P \breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle$ for all i, j.

Since $\breve{A} \subseteq_P \breve{A} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{A}_{ij}^a$ and $\lambda_{ij}^a \le \lambda_{ij}^a$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned}\breve{A} \wedge_P \breve{A} &= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{A}_{ij}^{a^+}\}], \min\{\lambda_{ij}^a, \lambda_{ij}^a\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \breve{A}.\end{aligned}$$

(ii) $\breve{A} \wedge_P \breve{B} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle$ for all i, j.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \le \mu_{ij}^b$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned}\breve{A} \wedge_P \breve{B} &= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \\ &= \breve{A}.\end{aligned}$$

(iii) $(\breve{A} \wedge_P \breve{B}) \wedge_P \breve{C} = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \right) \wedge_P \breve{C}$.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \le \mu_{ij}^b$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned}(\breve{A} \wedge_P \breve{B}) \wedge_P \breve{C} &= \left(\left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right) \wedge_P \breve{C} \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_P \breve{C} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle.\end{aligned}$$

Since $\breve{A} \subseteq_P \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \le \gamma_{ij}^c$ for all i, j.

Again by Definition 3.5(b),

$$\begin{aligned}(\breve{A} \wedge_P \breve{B}) \wedge_P \breve{C} &= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \\ &= \breve{A}.\end{aligned}$$

Similarly, $\breve{A} \wedge_P (\breve{B} \wedge_P \breve{C}) = \breve{A}$.

Hence, $(\breve{A} \wedge_P \breve{B}) \wedge_P \breve{C} = \breve{A} \wedge_P (\breve{B} \wedge_P \breve{C}) = \breve{A}$.

Proposition 3.12 *Let* $\breve{A}, \breve{B}, \breve{C} \in ECSM_{m\times n}^R$. *Then*

(i) $\breve{A} \wedge_R \breve{A} = \breve{A}$.

(ii) $\breve{A} \wedge_R \breve{B} = \breve{A}$.

(iii) $(\breve{A} \wedge_R \breve{B}) \wedge_R \breve{C} = \breve{A} \wedge_R (\breve{B} \wedge_R \breve{C}) = \breve{A}$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \in ECSM_{m\times n}^R$.

Then (i) $\breve{A} \wedge_R \breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle$ for all i, j.

Since $\breve{A} \subseteq_R \breve{A} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{A}_{ij}^a$ and $\lambda_{ij}^a \ge \lambda_{ij}^a$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned}\breve{A} \wedge_R \breve{A} &= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{A}_{ij}^{a^+}\}], \max\{\lambda_{ij}^a, \lambda_{ij}^a\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \breve{A}.\end{aligned}$$

(ii) $\breve{A} \wedge_R \breve{B} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle$ for all i, j.

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned}\breve{A} \wedge_R \breve{B} &= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \\ &= \breve{A}.\end{aligned}$$

(iii) $(\breve{A} \wedge_R \breve{B}) \wedge_R \breve{C} = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \right) \wedge_R \breve{C}$.

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned}(\breve{A} \wedge_R \breve{B}) \wedge_R \breve{C} &= \left(\left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right) \wedge_R \breve{C} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \breve{C} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle.\end{aligned}$$

Since $\breve{A} \subseteq_R \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \ge \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$\left(\breve{A} \wedge_R \breve{B}\right) \wedge_R \breve{C} = \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle$$
$$= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$$
$$= \breve{A}.$$

Similarly, $\breve{A} \wedge_R \left(\breve{B} \wedge_R \breve{C}\right) = \breve{A}$.

Hence, $\left(\breve{A} \wedge_R \breve{B}\right) \wedge_R \breve{C} = \breve{A} \wedge_R \left(\breve{B} \wedge_R \breve{C}\right) = \breve{A}$.

Theorem 3.13 Let $\breve{A}_1, \breve{A}_2, \breve{A}_3, \ldots, \breve{A}_k \in ECSM_{m\times n}^P$. Since $\breve{A}_1 \subseteq_P \breve{A}_2 \subseteq_P \breve{A}_3 \subseteq_P \cdots \subseteq_P \breve{A}_k$.

Then $\breve{A}_1 \vee_P \breve{A}_2 \vee_P \breve{A}_3 \vee_P \ldots \vee_P \breve{A}_k = \breve{A}_k$.

Proof. Let

$$\breve{A}_1 = \left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^a \right\rangle, \breve{A}_2 = \left\langle [\tilde{A}_{2_{ij}}^{a^-}, \tilde{A}_{2_{ij}}^{a^+}], \lambda_{2_{ij}}^a \right\rangle, \ \breve{A}_3 = \left\langle [\tilde{A}_{3_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^+}], \lambda_{3_{ij}}^a \right\rangle, \ldots,$$
$$\breve{A}_k = \left\langle [\tilde{A}_{k_{ij}}^{a^-}, \tilde{A}_{k_{ij}}^{a^+}], \lambda_{k_{ij}}^a \right\rangle \in ECSM_{m\times n}^P.$$

Then, $\breve{A}_1 \vee_P \breve{A}_2 \vee_P \breve{A}_3 \vee_P \ldots \vee_P \breve{A}_k = \left(\breve{A}_1 \vee_P \breve{A}_2\right) \vee_P \breve{A}_3 \vee_P \ldots \vee_P \breve{A}_k$.

Since, $\breve{A}_1 \subseteq_P \breve{A}_2 \Leftrightarrow \tilde{A}_{1_{ij}}^a \le \tilde{A}_{2_{ij}}^a$ and $\lambda_{1_{ij}}^a \le \lambda_{2_{ij}}^a$ for all i, j.

By Definition 3.5(a),

$$= \left(\left\langle [\max\{\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{2_{ij}}^{a^-}\}, \max\{\tilde{A}_{1_{ij}}^{a^+}, \tilde{A}_{2_{ij}}^{a^+}\}], \max\{\lambda_{1_{ij}}^a, \lambda_{2_{ij}}^a\} \right\rangle\right) \vee_P \breve{A}_3 \vee_P \ldots \vee_P \breve{A}_k$$
$$= \left(\left\langle [\tilde{A}_{2_{ij}}^{a^-}, \tilde{A}_{2_{ij}}^{a^+}], \lambda_{2_{ij}}^a \right\rangle\right) \vee_P \breve{A}_3 \vee_P \ldots \vee_P \breve{A}_k$$
$$= \left(\left\langle [\tilde{A}_{2_{ij}}^{a^-}, \tilde{A}_{2_{ij}}^{a^+}], \lambda_{2_{ij}}^a \right\rangle \vee_P \left\langle [\tilde{A}_{3_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^+}], \lambda_{3_{ij}}^a \right\rangle\right) \vee_P \breve{A}_4 \vee_P \ldots \vee_P \breve{A}_k.$$

Since, $\breve{A}_2 \subseteq_P \breve{A}_3 \Leftrightarrow \tilde{A}_{2_{ij}}^a \le \tilde{A}_{3_{ij}}^a$ and $\lambda_{2_{ij}}^a \le \lambda_{3_{ij}}^a$ for all i, j.

Again by Definition 3.5(a),

$$= \left(\left\langle [\max\{\tilde{A}_{2_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^-}\}, \max\{\tilde{A}_{2_{ij}}^{a^+}, \tilde{A}_{3_{ij}}^{a^+}\}], \max\{\lambda_{2_{ij}}^a, \lambda_{3_{ij}}^a\} \right\rangle\right) \vee_P \breve{A}_3 \vee_P \ldots \vee_P \breve{A}_k$$
$$= \left(\left\langle [\tilde{A}_{3_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^+}], \lambda_{3_{ij}}^a \right\rangle\right) \vee_P \breve{A}_4 \vee_P \ldots \vee_P \breve{A}_k$$
$$= \left(\left\langle [\tilde{A}_{3_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^+}], \lambda_{3_{ij}}^a \right\rangle \vee_P \left\langle [\tilde{A}_{4_{ij}}^{a^-}, \tilde{A}_{4_{ij}}^{a^+}], \lambda_{4_{ij}}^a \right\rangle\right) \vee_P \breve{A}_5 \vee_P \ldots \vee_P \breve{A}_k.$$

Continuing the same process we get,

$$= \left(\left\langle [\tilde{A}_{(k-1)_{ij}}^{a^-}, \tilde{A}_{(k-1)_{ij}}^{a^+}], \lambda_{(k-1)_{ij}}^a \right\rangle \vee_P \left\langle [\tilde{A}_{k_{ij}}^{a^-}, \tilde{A}_{k_{ij}}^{a^+}], \lambda_{k_{ij}}^a \right\rangle\right).$$

Since, $\breve{A}_{(k-1)} \subseteq_P \breve{A}_k \Leftrightarrow \tilde{A}_{(k-1)_{ij}}^a \le \tilde{A}_{k_{ij}}^a$ and $\lambda_{(k-1)_{ij}}^a \le \lambda_{k_{ij}}^a$ for all i, j.

Again by Definition 3.5(a),

$$= \left(\left\langle [\max\{\tilde{A}_{(k-1)_{ij}}^{a^-}, \tilde{A}_{k_{ij}}^{a^-}\}, \max\{\tilde{A}_{(k-1)_{ij}}^{a^+}, \tilde{A}_{k_{ij}}^{a^+}\}], \max\{\lambda_{(k-1)_{ij}}^a, \lambda_{k_{ij}}^a\} \right\rangle\right) \text{ for all i, j}$$
$$= \left(\left\langle [\tilde{A}_{k_{ij}}^{a^-}, \tilde{A}_{k_{ij}}^{a^+}], \lambda_{k_{ij}}^a \right\rangle\right)$$
$$= \left(\left\langle \tilde{A}_{k_{ij}}^a, \lambda_{k_{ij}}^a \right\rangle\right)$$
$$= \breve{A}_k.$$

Theorem 3.14 Let $\breve{A}_1, \breve{A}_2, \breve{A}_3, \ldots, \breve{A}_k \in ECSM^{R}_{m\times n}$. Since $\breve{A}_1 \subseteq_R \breve{A}_2 \subseteq_R \breve{A}_3 \subseteq_R \ldots \subseteq_R \breve{A}_k$.

Then $\breve{A}_1 \vee_R \breve{A}_2 \vee_R \breve{A}_3 \vee_R \ldots \vee_R \breve{A}_k = \breve{A}_k$.

Proof. Let

$$\breve{A}_1 = \left\langle [\tilde{A}^{a^-}_{1_{ij}}, \tilde{A}^{a^+}_{1_{ij}}], \lambda^a_{1_{ij}} \right\rangle, \breve{A}_2 = \left\langle [\tilde{A}^{a^-}_{2_{ij}}, \tilde{A}^{a^+}_{2_{ij}}], \lambda^a_{2_{ij}} \right\rangle, \breve{A}_3 = \left\langle [\tilde{A}^{a^-}_{3_{ij}}, \tilde{A}^{a^+}_{3_{ij}}], \lambda^a_{3_{ij}} \right\rangle, \ldots,$$

$$\breve{A}_k = \left\langle [\tilde{A}^{a^-}_{k_{ij}}, \tilde{A}^{a^+}_{k_{ij}}], \lambda^a_{k_{ij}} \right\rangle \in ECSM^{R}_{m\times n}.$$

Then, $\breve{A}_1 \vee_R \breve{A}_2 \vee_R \breve{A}_3 \vee_R \ldots \vee_R \breve{A}_k = \left(\breve{A}_1 \vee_R \breve{A}_2\right) \vee_R \breve{A}_3 \vee_R \ldots \vee_R \breve{A}_k$.

Since, $\breve{A}_1 \subseteq_R \breve{A}_2 \Leftrightarrow \tilde{A}^a_{1_{ij}} \le \tilde{A}^a_{2_{ij}}$ and $\lambda^a_{1_{ij}} \ge \lambda^a_{2_{ij}}$ for all i, j.

By Definition 3.7(a),

$$= \left(\left\langle [\max\{\tilde{A}^{a^-}_{1_{ij}}, \tilde{A}^{a^-}_{2_{ij}}\}, \max\{\tilde{A}^{a^+}_{1_{ij}}, \tilde{A}^{a^+}_{2_{ij}}\}], \min\{\lambda^a_{1_{ij}}, \lambda^a_{2_{ij}}\} \right\rangle\right) \vee_R \breve{A}_3 \vee_R \ldots \vee_R \breve{A}_k \text{ for all i, j}$$

$$= \left(\left\langle [\tilde{A}^{a^-}_{2_{ij}}, \tilde{A}^{a^+}_{2_{ij}}], \lambda^a_{2_{ij}} \right\rangle\right) \vee_R \breve{A}_3 \vee_R \ldots \vee_R \breve{A}_k$$

$$= \left(\left\langle [\tilde{A}^{a^-}_{2_{ij}}, \tilde{A}^{a^+}_{2_{ij}}], \lambda^a_{2_{ij}} \right\rangle \vee_R \left\langle [\tilde{A}^{a^-}_{3_{ij}}, \tilde{A}^{a^+}_{3_{ij}}], \lambda^a_{3_{ij}} \right\rangle\right) \vee_R \breve{A}_4 \vee_R \ldots \vee_R \breve{A}_k.$$

Since, $\breve{A}_2 \subseteq_R \breve{A}_3 \Leftrightarrow \tilde{A}^a_{2_{ij}} \le \tilde{A}^a_{3_{ij}}$ and $\lambda^a_{2_{ij}} \ge \lambda^a_{3_{ij}}$ for all i, j.

Again by Definition 3.7(a),

$$= \left(\left\langle [\max\{\tilde{A}^{a^-}_{2_{ij}}, \tilde{A}^{a^-}_{3_{ij}}\}, \max\{\tilde{A}^{a^+}_{2_{ij}}, \tilde{A}^{a^+}_{3_{ij}}\}], \min\{\lambda^a_{2_{ij}}, \lambda^a_{3_{ij}}\} \right\rangle\right) \vee_R \breve{A}_3 \vee_R \ldots \vee_R \breve{A}_k \text{ for all i, j}$$

$$= \left(\left\langle [\tilde{A}^{a^-}_{3_{ij}}, \tilde{A}^{a^+}_{3_{ij}}], \lambda^a_{3_{ij}} \right\rangle\right) \vee_R \breve{A}_4 \vee_R \ldots \vee_R \breve{A}_k$$

$$= \left(\left\langle [\tilde{A}^{a^-}_{3_{ij}}, \tilde{A}^{a^+}_{3_{ij}}], \lambda^a_{3_{ij}} \right\rangle \vee_R \left\langle [\tilde{A}^{a^-}_{4_{ij}}, \tilde{A}^{a^+}_{4_{ij}}], \lambda^a_{4_{ij}} \right\rangle\right) \vee_R \breve{A}_5 \vee_R \ldots \vee_R \breve{A}_k.$$

Continuing the same process we get,

$$= \left(\left\langle [\tilde{A}^{a^-}_{(k-1)_{ij}}, \tilde{A}^{a^+}_{(k-1)_{ij}}], \lambda^a_{(k-1)_{ij}} \right\rangle \vee_R \left\langle [\tilde{A}^{a^-}_{k_{ij}}, \tilde{A}^{a^+}_{k_{ij}}], \lambda^a_{k_{ij}} \right\rangle\right).$$

Since, $\breve{A}_{(k-1)} \subseteq_R \breve{A}_k \Leftrightarrow \tilde{A}^a_{(k-1)_{ij}} \le \tilde{A}^a_{k_{ij}}$ and $\lambda^a_{(k-1)_{ij}} \ge \lambda^a_{k_{ij}}$ for all i, j.

Again by Definition 3.7(a),

$$= \left(\left\langle [\max\{\tilde{A}^{a^-}_{(k-1)_{ij}}, \tilde{A}^{a^-}_{k_{ij}}\}, \max\{\tilde{A}^{a^+}_{(k-1)_{ij}}, \tilde{A}^{a^+}_{k_{ij}}\}], \min\{\lambda^a_{(k-1)_{ij}}, \lambda^a_{k_{ij}}\} \right\rangle\right) \text{ for all i, j}$$

$$= \left(\left\langle [\tilde{A}^{a^-}_{k_{ij}}, \tilde{A}^{a^+}_{k_{ij}}], \lambda^a_{k_{ij}} \right\rangle\right)$$

$$= \left(\left\langle \tilde{A}^a_{k_{ij}}, \lambda^a_{k_{ij}} \right\rangle\right)$$

$$= \breve{A}_k.$$

Theorem 3.15 Let $\breve{A}_1, \breve{A}_2, \breve{A}_3, \ldots, \breve{A}_k \in ECSM^{P}_{m\times n}$. Since $\breve{A}_1 \subseteq_P \breve{A}_2 \subseteq_P \breve{A}_3 \subseteq_P \ldots \subseteq_P \breve{A}_k$.

Then $\breve{A}_1 \wedge_P \breve{A}_2 \wedge_P \breve{A}_3 \wedge_P \ldots \wedge_P \breve{A}_k = \breve{A}_1$.

Proof. Let

$$\breve{A}_1 = \left\langle [\tilde{A}^{a^-}_{1_{ij}}, \tilde{A}^{a^+}_{1_{ij}}], \lambda^a_{1_{ij}} \right\rangle, \breve{A}_2 = \left\langle [\tilde{A}^{a^-}_{2_{ij}}, \tilde{A}^{a^+}_{2_{ij}}], \lambda^a_{2_{ij}} \right\rangle, \breve{A}_3 = \left\langle [\tilde{A}^{a^-}_{3_{ij}}, \tilde{A}^{a^+}_{3_{ij}}], \lambda^a_{3_{ij}} \right\rangle, \ldots,$$

$$\breve{A}_k = \left\langle [\tilde{A}^{a^-}_{k_{ij}}, \tilde{A}^{a^+}_{k_{ij}}], \lambda^a_{k_{ij}} \right\rangle \in ECSM^{P}_{m\times n}.$$

Then, $\breve{A}_1 \wedge_P \breve{A}_2 \wedge_P \breve{A}_3 \wedge_P \ldots \wedge_P \breve{A}_k = \left(\breve{A}_1 \wedge_P \breve{A}_2\right) \wedge_P \breve{A}_3 \wedge_P \ldots \wedge_P \breve{A}_k$.

Since, $\breve{A}_1 \subseteq_P \breve{A}_2 \Leftrightarrow \tilde{A}^a_{1_{ij}} \le \tilde{A}^a_{2_{ij}}$ and $\lambda^a_{1_{ij}} \le \lambda^a_{2_{ij}}$ for all i, j.

By Definition 3.5(b),

$$= \left(\left\langle [\min\{\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{2_{ij}}^{a^-}\}, \min\{\tilde{A}_{1_{ij}}^{a^+}, \tilde{A}_{2_{ij}}^{a^+}\}], \min\{\lambda_{1_{ij}}^{a}, \lambda_{2_{ij}}^{a}\} \right\rangle \right) \wedge_P \breve{A}_3 \wedge_P \ldots \wedge_P \breve{A}_k \text{ for all i, j}$$

$$= \left(\left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle \right) \wedge_P \breve{A}_3 \wedge_P \ldots \wedge_P \breve{A}_k$$

$$= \left(\left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle \wedge_P \left\langle [\tilde{A}_{3_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^+}], \lambda_{3_{ij}}^{a} \right\rangle \right) \wedge_P \breve{A}_4 \wedge_P \ldots \wedge_P \breve{A}_k .$$

Since, $\breve{A}_1 \subseteq_P \breve{A}_3 \Leftrightarrow \tilde{A}_{1_{ij}}^{a} \leq \tilde{A}_{3_{ij}}^{a}$ and $\lambda_{1_{ij}}^{a} \leq \lambda_{3_{ij}}^{a}$ for all i, j.

Again by Definition 3.5(b),

$$= \left(\left\langle [\min\{\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^-}\}, \min\{\tilde{A}_{1_{ij}}^{a^+}, \tilde{A}_{3_{ij}}^{a^+}\}], \min\{\lambda_{1_{ij}}^{a}, \lambda_{3_{ij}}^{a}\} \right\rangle \right) \wedge_P \breve{A}_3 \wedge_P \ldots \wedge_P \breve{A}_k \text{ for all i, j}$$

$$= \left(\left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle \right) \wedge_P \breve{A}_4 \wedge_P \ldots \wedge_P \breve{A}_k$$

$$= \left(\left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle \wedge_P \left\langle [\tilde{A}_{4_{ij}}^{a^-}, \tilde{A}_{4_{ij}}^{a^+}], \lambda_{4_{ij}}^{a} \right\rangle \right) \wedge_P \breve{A}_5 \wedge_P \ldots \wedge_P \breve{A}_k .$$

Continuing the same process we get,

$$= \left(\left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle \wedge_P \left\langle [\tilde{A}_{k_{ij}}^{a^-}, \tilde{A}_{k_{ij}}^{a^+}], \lambda_{k_{ij}}^{a} \right\rangle \right).$$

Since, $\breve{A}_1 \subseteq_P \breve{A}_k \Leftrightarrow \tilde{A}_{1_{ij}}^{a} \leq \tilde{A}_{k_{ij}}^{a}$ and $\lambda_{1_{ij}}^{a} \leq \lambda_{kij}^{a}$ for all i, j.

Again by Definition 3.5(b),

$$= \left(\left\langle [\min\{\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{k_{ij}}^{a^-}\}, \min\{\tilde{A}_{1_{ij}}^{a^+}, \tilde{A}_{k_{ij}}^{a^+}\}], \min\{\lambda_{1_{ij}}^{a}, \lambda_{k_{ij}}^{a}\} \right\rangle \right) \text{for all i, j}$$

$$= \left(\left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle \right)$$

$$= \left(\left\langle \tilde{A}_{1_{ij}}^{a}, \lambda_{1_{ij}}^{a} \right\rangle \right)$$

$$= \breve{A}_1 .$$

Theorem 3.16 Let $\breve{A}_1, \breve{A}_2, \breve{A}_3, \ldots, \breve{A}_k \in ECSM_{m\times n}^{R}$. Since $\breve{A}_1 \subseteq_R \breve{A}_2 \subseteq_R \breve{A}_3 \subseteq_R \ldots \subseteq_R \breve{A}_k$.

Then $\breve{A}_1 \wedge_R \breve{A}_2 \wedge_R \breve{A}_3 \wedge_R \ldots \wedge_R \breve{A}_k = \breve{A}_1$.

Proof. Let

$$\breve{A}_1 = \left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle, \breve{A}_2 = \left\langle [\tilde{A}_{2_{ij}}^{a^-}, \tilde{A}_{2_{ij}}^{a^+}], \lambda_{2_{ij}}^{a} \right\rangle, \breve{A}_3 = \left\langle [\tilde{A}_{3_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^+}], \lambda_{3_{ij}}^{a} \right\rangle, \ldots,$$

$$\breve{A}_k = \left\langle [\tilde{A}_{k_{ij}}^{a^-}, \tilde{A}_{k_{ij}}^{a^+}], \lambda_{k_{ij}}^{a} \right\rangle \in ECSM_{m\times n}^{R}.$$

Then, $\breve{A}_1 \wedge_R \breve{A}_2 \wedge_R \breve{A}_3 \wedge_R \ldots \wedge_R \breve{A}_k = \left(\breve{A}_1 \wedge_R \breve{A}_2\right) \wedge_R \breve{A}_3 \wedge_R \ldots \wedge_R \breve{A}_k$.

Since, $\breve{A}_1 \subseteq_R \breve{A}_2 \Leftrightarrow \tilde{A}_{1_{ij}}^{a} \leq \tilde{A}_{2_{ij}}^{a}$ and $\lambda_{1_{ij}}^{a} \geq \lambda_{2_{ij}}^{a}$ for all i, j.

By Definition 3.7(b),

$$= \left(\left\langle [\min\{\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{2_{ij}}^{a^-}\}, \min\{\tilde{A}_{1_{ij}}^{a^+}, \tilde{A}_{2_{ij}}^{a^+}\}], \max\{\lambda_{1_{ij}}^{a}, \lambda_{2_{ij}}^{a}\} \right\rangle \right) \wedge_R \breve{A}_3 \wedge_R \ldots \wedge_R \breve{A}_k \text{ for all i, j}$$

$$= \left(\left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle \right) \wedge_R \breve{A}_3 \wedge_R \ldots \wedge_R \breve{A}_k$$

$$= \left(\left\langle [\tilde{A}_{1_{ij}}^{a^-}, \tilde{A}_{1_{ij}}^{a^+}], \lambda_{1_{ij}}^{a} \right\rangle \wedge_R \left\langle [\tilde{A}_{3_{ij}}^{a^-}, \tilde{A}_{3_{ij}}^{a^+}], \lambda_{3_{ij}}^{a} \right\rangle \right) \wedge_R \breve{A}_4 \wedge_R \ldots \wedge_R \breve{A}_k .$$

Since, $\breve{A}_1 \subseteq_R \breve{A}_3 \Leftrightarrow \tilde{A}_{1_{ij}}^{a} \leq \tilde{A}_{3_{ij}}^{a}$ and $\lambda_{1_{ij}}^{a} \geq \lambda_{3_{ij}}^{a}$ for all i, j.

Again by Definition 3.7(b),

$$=\left(\left\langle[\min\{\tilde{A}^{a^-}_{1_{ij}},\tilde{A}^{a^-}_{3_{ij}}\},\min\{\tilde{A}^{a^+}_{1_{ij}},\tilde{A}^{a^+}_{3_{ij}}\}],\max\{\lambda^a_{1_{ij}},\lambda^a_{3_{ij}}\}\right\rangle\right)\wedge_R \breve{A}_3 \wedge_R \ldots \wedge_R \breve{A}_k \text{ for all i, j}$$
$$=\left(\left\langle[\tilde{A}^{a^-}_{1_{ij}},\tilde{A}^{a^+}_{1_{ij}}],\lambda^a_{1_{ij}}\right\rangle\right)\wedge_R \breve{A}_4 \wedge_R \ldots \wedge_R \breve{A}_k$$
$$=\left(\left\langle[\tilde{A}^{a^-}_{1_{ij}},\tilde{A}^{a^+}_{1_{ij}}],\lambda^a_{1_{ij}}\right\rangle\wedge_R\left\langle[\tilde{A}^{a^-}_{4_{ij}},\tilde{A}^{a^+}_{4_{ij}}],\lambda^a_{4_{ij}}\right\rangle\right)\wedge_R \breve{A}_5 \wedge_R \ldots \wedge_R \breve{A}_k .$$

Continuing the same process we get,

$$=\left(\left\langle[\tilde{A}^{a^-}_{1_{ij}},\tilde{A}^{a^+}_{1_{ij}}],\lambda^a_{1_{ij}}\right\rangle\wedge_R\left\langle[\tilde{A}^{a^-}_{k_{ij}},\tilde{A}^{a^+}_{k_{ij}}],\lambda^a_{k_{ij}}\right\rangle\right).$$

Since, $\breve{A}_1 \subseteq_R \breve{A}_k \Leftrightarrow \tilde{A}^a_{1_{ij}} \leq \tilde{A}^a_{k_{ij}}$ and $\lambda^a_{1_{ij}} \geq \lambda_{k\,ij}^{\;a}$ for all i, j.

Again by Definition 3.7(b),

$$=\left(\left\langle[\min\{\tilde{A}^{a^-}_{1_{ij}},\tilde{A}^{a^-}_{k_{ij}}\},\min\{\tilde{A}^{a^+}_{1_{ij}},\tilde{A}^{a^+}_{k_{ij}}\}],\max\{\lambda^a_{1_{ij}},\lambda^a_{k_{ij}}\}\right\rangle\right)\text{for all i, j}$$
$$=\left(\left\langle[\tilde{A}^{a^-}_{1_{ij}},\tilde{A}^{a^+}_{1_{ij}}],\lambda^a_{1_{ij}}\right\rangle\right)$$
$$=\left(\left\langle\tilde{A}^a_{1_{ij}},\lambda^a_{1_{ij}}\right\rangle\right)$$
$$=\breve{A}_1 .$$

Theorem 3.17 Let $\breve{A},\breve{B},\breve{C} \in ECSM^P_{m\times n}$, then $\left(\breve{A} \vee_P \breve{B}\right)\wedge_P \breve{C} = \left(\breve{A} \wedge_P \breve{C}\right)\vee_P\left(\breve{B} \wedge_P \breve{C}\right) = \breve{B}$.

Proof. Let $\breve{A} = \left\langle[\tilde{A}^{a^-}_{ij},\tilde{A}^{a^+}_{ij}],\lambda^a_{ij}\right\rangle, \breve{B} = \left\langle[\tilde{B}^{b^-}_{ij},\tilde{B}^{b^+}_{ij}],\mu^b_{ij}\right\rangle, \breve{C} = \left\langle[\tilde{C}^{c^-}_{ij},\tilde{C}^{c^+}_{ij}],\gamma^c_{ij}\right\rangle \in ICSM^P_{m\times n}$.

Then, L.H.S= $\left(\breve{A} \vee_P \breve{B}\right)\wedge_P \breve{C} = \left(\left\langle[\tilde{A}^{a^-}_{ij},\tilde{A}^{a^+}_{ij}],\lambda^a_{ij}\right\rangle\vee_P\left\langle[\tilde{B}^{b^-}_{ij},\tilde{B}^{b^+}_{ij}],\mu^b_{ij}\right\rangle\right)\wedge_P \breve{C}$.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}^a_{ij} \leq \tilde{B}^b_{ij}$ and $\lambda^a_{ij} \leq \mu^b_{ij}$ for all i, j.

By Definition 3.5(a),

$$\left(\breve{A} \vee_P \breve{B}\right)\wedge_P \breve{C} = \left(\left\langle[\max\{\tilde{A}^{a^-}_{ij},\tilde{B}^{b^-}_{ij}\},\max\{\tilde{A}^{a^+}_{ij},\tilde{B}^{b^+}_{ij}\}],\max\{\lambda^a_{ij},\mu^b_{ij}\}\right\rangle\right)\wedge_P \breve{C} \text{ for all i, j}$$
$$=\left\langle[\tilde{B}^{b^-}_{ij},\tilde{B}^{b^+}_{ij}],\mu^b_{ij}\right\rangle\wedge_P \breve{C}$$
$$=\left\langle[\tilde{B}^{b^-}_{ij},\tilde{B}^{b^+}_{ij}],\mu^b_{ij}\right\rangle\wedge_P\left\langle[\tilde{C}^{c^-}_{ij},\tilde{C}^{c^+}_{ij}],\gamma^c_{ij}\right\rangle.$$

Since $\breve{B} \subseteq_P \breve{C} \Leftrightarrow \tilde{B}^b_{ij} \leq \tilde{C}^c_{ij}$ and $\mu^b_{ij} \leq \gamma^c_{ij}$ for all i, j.

By the Definition 3.5(b),

$$=\left\langle[\min\{\tilde{B}^{b^-}_{ij},\tilde{C}^{c^-}_{ij}\},\min\{\tilde{B}^{b^+}_{ij},\tilde{C}^{c^+}_{ij}\}],\min\{\mu^b_{ij},\gamma^c_{ij}\}\right\rangle \text{for all i, j}$$
$$=\left\langle[\tilde{B}^{b^-}_{ij},\tilde{B}^{b^+}_{ij}],\mu^b_{ij}\right\rangle$$
$$=\left\langle\tilde{B}^b_{ij},\mu^b_{ij}\right\rangle$$
$$=\breve{B} .$$

R.H.S= $\left(\breve{A} \wedge_P \breve{C}\right)\vee_P\left(\breve{B} \wedge_P \breve{C}\right)$.

Then, $\left(\breve{A} \wedge_P \breve{C}\right) = \left(\left\langle[\tilde{A}^{a^-}_{ij},\tilde{A}^{a^+}_{ij}],\lambda^a_{ij}\right\rangle\wedge_P\left\langle[\tilde{C}^{c^-}_{ij},\tilde{C}^{c^+}_{ij}],\gamma^c_{ij}\right\rangle\right)$.

Since $\breve{A} \subseteq_P \breve{C} \Leftrightarrow \tilde{A}^a_{ij} \leq \tilde{C}^c_{ij}$ and $\lambda^a_{ij} \leq \gamma^c_{ij}$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned}(\breve{A} \wedge_P \breve{C}) &= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j}\\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle\\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle.\end{aligned}$$

Then, $(\breve{B} \wedge_P \breve{C}) = \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \wedge_P \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \right)$.

Since $\breve{B} \subseteq_P \breve{C} \Leftrightarrow \tilde{B}_{ij}^b \le \tilde{C}_{ij}^c$ and $\mu_{ij}^b \le \gamma_{ij}^c$ for all i, j.

Again by Definition 3.5(b),

$$\begin{aligned}(\breve{B} \wedge_P \breve{C}) &= \left\langle [\min\{\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+}\}], \min\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all i, j}\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle.\\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle.\\ (\breve{A} \wedge_P \breve{C}) \vee_P (\breve{B} \wedge_P \breve{C}) &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_P \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle\\ &= \breve{A} \vee_P \breve{B}.\end{aligned}$$

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \le \mu_{ij}^b$ for all i, j.

By Definition 3.5(a),

$$\begin{aligned}(\breve{A} \wedge_P \breve{C}) \vee_P (\breve{B} \wedge_P \breve{C}) &= \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all i, j}\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\\ &= \breve{B}.\end{aligned}$$

Hence, $(\breve{A} \vee_P \breve{B}) \wedge_P \breve{C} = (\breve{A} \wedge_P \breve{C}) \vee_P (\breve{B} \wedge_P \breve{C}) = \breve{B}$.

Theorem 3.18 Let $\breve{A}, \breve{B}, \breve{C} \in ECSM_{m \times n}^R$, then $(\breve{A} \vee_R \breve{B}) \wedge_R \breve{C} = (\breve{A} \wedge_R \breve{C}) \vee_R (\breve{B} \wedge_R \breve{C}) = \breve{B}$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \in ICSM_{m \times n}^R$.

Then, L.H.S= $(\breve{A} \vee_R \breve{B}) \wedge_R \breve{C} = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \right) \wedge_R \breve{C}$.

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned}(\breve{A} \vee_R \breve{B}) \wedge_R \breve{C} &= \left(\left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \right) \wedge_R \breve{C} \text{ for all i, j}\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \wedge_R \breve{C}\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle.\end{aligned}$$

Since $\breve{B} \subseteq_R \breve{C} \Leftrightarrow \tilde{B}_{ij}^b \le \tilde{C}_{ij}^c$ and $\mu_{ij}^b \ge \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$= \left\langle [\min\{\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\mu_{ij}^{b}, \gamma_{ij}^{c}\} \right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle$$
$$= \left\langle \tilde{B}_{ij}^{b}, \mu_{ij}^{b} \right\rangle$$
$$= \breve{B}.$$

R.H.S= $(\breve{A} \wedge_R \breve{C}) \vee_R (\breve{B} \wedge_R \breve{C})$.

Then, $(\breve{A} \wedge_R \breve{C}) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^{c} \right\rangle \right)$.

Since $\breve{A} \subseteq_R \breve{C} \Leftrightarrow \tilde{A}_{ij}^{a} \le \tilde{C}_{ij}^{c}$ and $\lambda_{ij}^{a} \ge \gamma_{ij}^{c}$ for all i, j.

By Definition 3.7(b),

$$(\breve{A} \wedge_R \breve{C}) = \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\lambda_{ij}^{a}, \gamma_{ij}^{c}\} \right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle$$
$$= \left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle.$$

Then, $(\breve{B} \wedge_R \breve{C}) = \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^{c} \right\rangle \right)$.

Since $\breve{B} \subseteq_R \breve{C} \Leftrightarrow \tilde{B}_{ij}^{b} \le \tilde{C}_{ij}^{c}$ and $\mu_{ij}^{b} \ge \gamma_{ij}^{c}$ for all i, j.

Again by Definition 3.7(b),

$$(\breve{B} \wedge_R \breve{C}) = \left\langle [\min\{\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\mu_{ij}^{b}, \gamma_{ij}^{c}\} \right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle$$
$$= \left\langle \tilde{B}_{ij}^{b}, \mu_{ij}^{b} \right\rangle.$$
$$(\breve{A} \wedge_R \breve{C}) \vee_R (\breve{B} \wedge_R \breve{C}) = \left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle \vee_R \left\langle \tilde{B}_{ij}^{b}, \mu_{ij}^{b} \right\rangle$$
$$= \breve{A} \vee_R \breve{B}.$$

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^{a} \le \tilde{B}_{ij}^{b}$ and $\lambda_{ij}^{a} \ge \mu_{ij}^{b}$ for all i, j.

By the Definition 3.7(a),

$$(\breve{A} \wedge_R \breve{C}) \vee_R (\breve{B} \wedge_R \breve{C}) = \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle$$
$$= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle$$
$$= \breve{B}.$$

Hence, $(\breve{A} \vee_R \breve{B}) \wedge_R \breve{C} = (\breve{A} \wedge_R \breve{C}) \vee_R (\breve{B} \wedge_R \breve{C}) = \breve{B}$.

Theorem 3.19 Let $\breve{A}, \breve{B}, \breve{C} \in ECSM_{m \times n}^{P}$, then

$(\breve{A} \wedge_P \breve{B}) \vee_P \breve{C} = (\breve{A} \vee_P \breve{C}) \wedge_P (\breve{B} \vee_P \breve{C}) = \breve{C}$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^{c} \right\rangle \in ICSM_{m \times n}^{P}$.

Then, L.H.S= $(\breve{A} \wedge_P \breve{B}) \vee_P \breve{C} = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle \wedge_P \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle \right) \vee_P \breve{C}$.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^{a} \le \tilde{B}_{ij}^{b}$ and $\lambda_{ij}^{a} \le \mu_{ij}^{b}$ for all i, j.

By the Definition 3.5(b),

$$\left(\breve{A} \wedge_P \breve{B}\right)\vee_P \breve{C} = \left(\left\langle [\min\{\tilde{A}_{ij}^{a^-},\tilde{B}_{ij}^{b^-}\},\min\{\tilde{A}_{ij}^{a^+},\tilde{B}_{ij}^{b^+}\}],\min\{\lambda_{ij}^a,\mu_{ij}^b\}\right\rangle\right)\vee_P \breve{C} \text{ for all i, j}$$
$$= \left\langle [\tilde{A}_{ij}^{a^-},\tilde{A}_{ij}^{a^+}],\lambda_{ij}^a\right\rangle \vee_P \breve{C}$$
$$= \left\langle [\tilde{A}_{ij}^{a^-},\tilde{A}_{ij}^{a^+}],\lambda_{ij}^a\right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c^-},\tilde{C}_{ij}^{c^+}],\gamma_{ij}^c\right\rangle.$$

Since $\breve{A} \subseteq_P \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \le \gamma_{ij}^c$ for all i, j.

By Definition 3.5(b),

$$= \left\langle [\max\{\tilde{A}_{ij}^{a^-},\tilde{C}_{ij}^{c^-}\},\min\{\tilde{A}_{ij}^{a^+},\tilde{C}_{ij}^{c^+}\}],\min\{\lambda_{ij}^a,\gamma_{ij}^c\}\right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{C}_{ij}^{c^-},\tilde{C}_{ij}^{c^+}],\gamma_{ij}^c\right\rangle$$
$$= \left\langle \tilde{C}_{ij}^c,\gamma_{ij}^c\right\rangle$$
$$= \breve{C}.$$

R.H.S= $\left(\breve{A} \vee_P \breve{C}\right)\wedge_P \left(\breve{B} \vee_P \breve{C}\right)$.

Then, $\left(\breve{A} \vee_P \breve{C}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-},\tilde{A}_{ij}^{a^+}],\lambda_{ij}^a\right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c^-},\tilde{C}_{ij}^{c^+}],\gamma_{ij}^c\right\rangle\right)$.

Since $\breve{A} \subseteq_P \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \le \gamma_{ij}^c$ for all i, j.

By Definition 3.5(a),

$$\left(\breve{A} \vee_P \breve{C}\right) = \left\langle [\max\{\tilde{A}_{ij}^{a^-},\tilde{C}_{ij}^{c^-}\},\max\{\tilde{A}_{ij}^{a^+},\tilde{C}_{ij}^{c^+}\}],\max\{\lambda_{ij}^a,\gamma_{ij}^c\}\right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{C}_{ij}^{c^-},\tilde{C}_{ij}^{c^+}],\gamma_{ij}^c\right\rangle$$
$$= \left\langle \tilde{C}_{ij}^c,\gamma_{ij}^c\right\rangle.$$

Then, $\left(\breve{B} \vee_P \breve{C}\right) = \left(\left\langle [\tilde{B}_{ij}^{b^-},\tilde{B}_{ij}^{b^+}],\mu_{ij}^b\right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c^-},\tilde{C}_{ij}^{c^+}],\gamma_{ij}^c\right\rangle\right)$.

Since $\breve{B} \subseteq_P \breve{C} \Leftrightarrow \tilde{B}_{ij}^b \le \tilde{C}_{ij}^c$ and $\mu_{ij}^b \le \gamma_{ij}^c$ for all i, j.

Again by Definition 3.5(a),

$$\left(\breve{B} \vee_P \breve{C}\right) = \left\langle [\max\{\tilde{B}_{ij}^{b^-},\tilde{C}_{ij}^{c^-}\},\max\{\tilde{B}_{ij}^{b^+},\tilde{C}_{ij}^{c^+}\}],\max\{\mu_{ij}^b,\gamma_{ij}^c\}\right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{C}_{ij}^{c^-},\tilde{C}_{ij}^{c^+}],\gamma_{ij}^c\right\rangle$$
$$= \left\langle \tilde{C}_{ij}^c,\gamma_{ij}^c\right\rangle$$
$$\left(\breve{A} \vee_P \breve{C}\right)\wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \left\langle \tilde{C}_{ij}^c,\gamma_{ij}^c\right\rangle \wedge_P \left\langle \tilde{C}_{ij}^c,\gamma_{ij}^c\right\rangle$$
$$= \breve{C} \wedge_P \breve{C}.$$

By Proposition 3.11(i),
$\left(\breve{A} \vee_P \breve{C}\right)\wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \breve{C}$ .Hence, $\left(\breve{A} \wedge_P \breve{B}\right)\vee_P \breve{C} = \left(\breve{A} \vee_P \breve{C}\right)\wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \breve{C}$.

Theorem 3.20 Let $\breve{A},\breve{B},\breve{C} \in ECSM_{m\times n}^R$, then
$\left(\breve{A} \wedge_R \breve{B}\right)\vee_R \breve{C} = \left(\breve{A} \vee_R \breve{C}\right)\wedge_R \left(\breve{B} \vee_R \breve{C}\right) = \breve{C}$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-},\tilde{A}_{ij}^{a^+}],\lambda_{ij}^a\right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-},\tilde{B}_{ij}^{b^+}],\mu_{ij}^b\right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-},\tilde{C}_{ij}^{c^+}],\gamma_{ij}^c\right\rangle \in ECSM_{m\times n}^R$.

Then, L.H.S= $\left(\breve{A} \wedge_R \breve{B}\right)\vee_R \breve{C} = \left(\left\langle [\tilde{A}_{ij}^{a^-},\tilde{A}_{ij}^{a^+}],\lambda_{ij}^a\right\rangle \wedge_R \left\langle [\tilde{B}_{ij}^{b^-},\tilde{B}_{ij}^{b^+}],\mu_{ij}^b\right\rangle\right)\vee_R \breve{C}$.

Since $A^{(} \subseteq_R B^{(} \Leftrightarrow \tilde{A}^a_{ij} \le \tilde{B}^b_{ij}$ and $\lambda^a_{ij} \ge \mu^b_{ij}$ for all i, j.

By Definition 3.7(b),

$$\left(A^{(} \wedge_R B^{(}\right) \vee_R C^{(} = \left(\left\langle [\min\{\tilde{A}^{a^-}_{ij}, \tilde{B}^{b^-}_{ij}\}, \min\{\tilde{A}^{a^+}_{ij}, \tilde{B}^{b^+}_{ij}\}], \max\{\lambda^a_{ij}, \mu^b_{ij}\}\right\rangle\right) \vee_R C^{(} \text{ for all i, j}$$
$$= \left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij}\right\rangle \vee_R C^{(}$$
$$= \left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij}\right\rangle \vee_R \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij}\right\rangle.$$

Since $A^{(} \subseteq_R C^{(} \Leftrightarrow \tilde{A}^a_{ij} \le \tilde{C}^c_{ij}$ and $\lambda^a_{ij} \ge \gamma^c_{ij}$ for all i, j.

By Definition 3.7(a),

$$= \left\langle [\max\{\tilde{A}^{a^-}_{ij}, \tilde{C}^{c^-}_{ij}\}, \max\{\tilde{A}^{a^+}_{ij}, \tilde{C}^{c^+}_{ij}\}], \min\{\lambda^a_{ij}, \gamma^c_{ij}\}\right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij}\right\rangle$$
$$= \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij}\right\rangle$$
$$= C^{(}.$$

R.H.S= $\left(A^{(} \vee_R C^{(}\right) \wedge_R \left(B^{(} \vee_R C^{(}\right)$.

Then, $\left(A^{(} \vee_R C^{(}\right) = \left(\left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij}\right\rangle \vee_R \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij}\right\rangle\right).$

Since $A^{(} \subseteq_R C^{(} \Leftrightarrow \tilde{A}^a_{ij} \le \tilde{C}^c_{ij}$ and $\lambda^a_{ij} \ge \gamma^c_{ij}$ for all i, j.

By Definition 3.7(a),

$$\left(A^{(} \vee_R C^{(}\right) = \left\langle [\max\{\tilde{A}^{a^-}_{ij}, \tilde{C}^{c^-}_{ij}\}, \max\{\tilde{A}^{a^+}_{ij}, \tilde{C}^{c^+}_{ij}\}], \min\{\lambda^a_{ij}, \gamma^c_{ij}\}\right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij}\right\rangle$$
$$= \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij}\right\rangle.$$

Then, $\left(B^{(} \vee_R C^{(}\right) = \left(\left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij}\right\rangle \vee_R \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij}\right\rangle\right).$

Since $B^{(} \subseteq_R C^{(} \Leftrightarrow \tilde{B}^b_{ij} \le \tilde{C}^c_{ij}$ and $\mu^b_{ij} \ge \gamma^c_{ij}$ for all i, j.

Again by Definition 3.7(a),

$$\left(B^{(} \vee_R C^{(}\right) = \left\langle [\max\{\tilde{B}^{b^-}_{ij}, \tilde{C}^{c^-}_{ij}\}, \max\{\tilde{B}^{b^+}_{ij}, \tilde{C}^{c^+}_{ij}\}], \min\{\mu^b_{ij}, \gamma^c_{ij}\}\right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij}\right\rangle$$
$$= \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij}\right\rangle.$$
$$\left(A^{(} \vee_R C^{(}\right) \wedge_R \left(B^{(} \vee_R C^{(}\right) = \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij}\right\rangle \wedge_R \left\langle \tilde{C}^c_{ij}, \gamma^c_{ij}\right\rangle$$
$$= C^{(} \wedge_R C^{(}.$$

By Proposition 3.12(i), $\left(A^{(} \vee_R C^{(}\right) \wedge_R \left(B^{(} \vee_R C^{(}\right) = C^{(}$.

Hence, $\left(A^{(} \wedge_R B^{(}\right) \vee_R C^{(} = \left(A^{(} \vee_R C^{(}\right) \wedge_R \left(B^{(} \vee_R C^{(}\right) = C^{(}$.

Theorem 3.21 Let $A^{(}, B^{(}, C^{(} \in ECSM^P_{m\times n}$, then

$$A^{(} \vee_P \left(B^{(} \wedge_P C^{(}\right) = \left(A^{(} \vee_P B^{(}\right) \wedge_P \left(A^{(} \vee_P C^{(}\right) = B^{(}.$$

Proof. Let $A^{(} = \left\langle [\tilde{A}^{a^-}_{ij}, \tilde{A}^{a^+}_{ij}], \lambda^a_{ij}\right\rangle, B^{(} = \left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij}\right\rangle, C^{(} = \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij}\right\rangle \in ECSM^P_{m\times n}$.

Then, L.H.S= $A^{(} \vee_P \left(B^{(} \wedge_P C^{(}\right) = A^{(} \vee_P \left(\left\langle [\tilde{B}^{b^-}_{ij}, \tilde{B}^{b^+}_{ij}], \mu^b_{ij}\right\rangle \wedge_P \left\langle [\tilde{C}^{c^-}_{ij}, \tilde{C}^{c^+}_{ij}], \gamma^c_{ij}\right\rangle\right).$

Since $\breve{B} \subseteq_P \breve{C} \Leftrightarrow \tilde{B}_{ij}^{b} \leq \tilde{C}_{ij}^{c}$ and $\mu_{ij}^{b} \leq \gamma_{ij}^{c}$ for all i, j.

By Definition 3.5(b),

$$\breve{A} \vee_P \left(\breve{B} \wedge_P \breve{C}\right) = \breve{A} \vee_P \left(\left\langle [\min(\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}), \min(\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+})], \min(\mu_{ij}^{b}, \gamma_{ij}^{c}) \right\rangle\right) \text{ for all i, j}$$

$$= \breve{A} \vee_P \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle$$

$$= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle \vee_P \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle.$$

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^{a} \leq \tilde{B}_{ij}^{b}$ and $\lambda_{ij}^{a} \leq \mu_{ij}^{b}$ for all i, j.

By Definition 3.5(a),

$$= \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \right\rangle \text{ for all i, j}$$

$$= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle$$

$$= \left\langle \tilde{B}_{ij}^{b}, \mu_{ij}^{b} \right\rangle$$

$$= \breve{B}.$$

R.H.S $= \left(\breve{A} \vee_P \breve{B}\right) \wedge_P \left(\breve{A} \vee_P \breve{C}\right).$

Then, $\left(\breve{A} \vee_P \breve{B}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle \vee_P \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle\right).$

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^{a} \leq \tilde{B}_{ij}^{b}$ and $\lambda_{ij}^{a} \leq \mu_{ij}^{b}$ for all i, j.

By Definition 3.5(a),

$$\left(\breve{A} \vee_P \breve{B}\right) = \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \right\rangle \text{ for all i, j}$$

$$= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^{b} \right\rangle$$

$$= \left\langle \tilde{B}_{ij}^{b}, \mu_{ij}^{b} \right\rangle.$$

Then, $\left(\breve{A} \vee_P \breve{C}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^{a} \right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^{c} \right\rangle\right).$

Since $\breve{A} \subseteq_P \breve{C} \Leftrightarrow \tilde{A}_{ij}^{a} \leq \tilde{C}_{ij}^{c}$ and $\lambda_{ij}^{a} \leq \gamma_{ij}^{c}$ for all i, j.

Again by Definition 3.5(a),

$$\left(\breve{A} \vee_P \breve{C}\right) = \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\lambda_{ij}^{a}, \gamma_{ij}^{c}\} \right\rangle \text{ for all i, j}$$

$$= \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^{c} \right\rangle$$

$$= \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$$

$$\left(\breve{A} \vee_P \breve{B}\right) \wedge_P \left(\breve{A} \wedge_P \breve{C}\right) = \left\langle \tilde{B}_{ij}^{b}, \mu_{ij}^{b} \right\rangle \wedge_P \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle$$

$$= \breve{B} \wedge_P \breve{C}.$$

Since $\breve{B} \subseteq_P \breve{C} \Leftrightarrow \tilde{B}_{ij}^{b} \leq \tilde{C}_{ij}^{c}$ and $\mu_{ij}^{b} \leq \gamma_{ij}^{c}$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned}\left(\breve{A} \vee_P \breve{B}\right)\wedge_P\left(\breve{A} \wedge_P \breve{C}\right) &= \left\langle [\min\{\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+}\}], \min\{\mu_{ij}^b, \gamma_{ij}^c\}\right\rangle \text{ for all i, j}\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle\\ &= \breve{B}.\end{aligned}$$

Hence, $\breve{A} \vee_P \left(\breve{B} \wedge_P \breve{C}\right) = \left(\breve{A} \vee_P \breve{B}\right)\wedge_P\left(\breve{A} \vee_P \breve{C}\right) = \breve{B}$.

Theorem 3.22 Let $\breve{A}, \breve{B}, \breve{C} \in ECSM_{m\times n}^R$, then

$\breve{A} \vee_R \left(\breve{B} \wedge_R \breve{C}\right) = \left(\breve{A} \vee_R \breve{B}\right)\wedge_R\left(\breve{A} \vee_R \breve{C}\right) = \breve{B}$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a\right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c\right\rangle \in ECSM_{m\times n}^R$.

Then, L.H.S= $\breve{A} \vee_R \left(\breve{B} \wedge_R \breve{C}\right) = \breve{A} \vee_R \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c\right\rangle\right)$.

Since $\breve{B} \subseteq_R \breve{C} \Leftrightarrow \tilde{B}_{ij}^b \le \tilde{C}_{ij}^c$ and $\mu_{ij}^b \ge \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned}\breve{A} \vee_R \left(\breve{B} \wedge_R \breve{C}\right) &= \breve{A} \vee_R \left(\left\langle [\min(\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}), \min(\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+})], \max(\mu_{ij}^b, \gamma_{ij}^c)\right\rangle\right) \text{ for all i, j}\\ &= \breve{A} \vee_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle\\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a\right\rangle \vee_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle.\end{aligned}$$

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned}&= \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\}\right\rangle \text{ for all i, j}\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle\\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b\right\rangle\\ &= \breve{B}.\end{aligned}$$

R.H.S= $\left(\breve{A} \vee_R \breve{B}\right)\wedge_R\left(\breve{A} \vee_R \breve{C}\right)$.

Then, $\left(\breve{A} \vee_R \breve{B}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a\right\rangle \vee_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle\right)$.

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned}\left(\breve{A} \vee_R \breve{B}\right) &= \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\}\right\rangle \text{ for all i, j}\\ &= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b\right\rangle\\ &= \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b\right\rangle.\end{aligned}$$

Then, $\left(\breve{A} \vee_R \breve{C}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a\right\rangle \vee_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c\right\rangle\right)$.

Since $\breve{A} \subseteq_R \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \ge \gamma_{ij}^c$ for all i, j.

Again by Definition 3.7(a),

$$\left(\breve{A} \vee_R \breve{C}\right) = \left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j}$$

$$= \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle$$

$$= \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$$

$$\left(\breve{A} \vee_R \breve{B}\right) \wedge_R \left(\breve{A} \wedge_R \breve{C}\right) = \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle \wedge_R \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$$

$$= \breve{B} \wedge_R \breve{C}.$$

Since $\breve{B} \subseteq_R \breve{C} \Leftrightarrow \tilde{B}_{ij}^b \le \tilde{C}_{ij}^c$ and $\mu_{ij}^b \ge \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$\left(\breve{A} \vee_R \breve{B}\right) \wedge_R \left(\breve{A} \wedge_R \breve{C}\right) = \left\langle [\min\{\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle \text{ for all i, j}$$

$$= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle$$

$$= \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle$$

$$= \breve{B}.$$

Hence, $\breve{A} \vee_R \left(\breve{B} \wedge_R \breve{C}\right) = \left(\breve{A} \vee_R \breve{B}\right) \wedge_R \left(\breve{A} \vee_R \breve{C}\right) = \breve{B}$.

Theorem 3.23 Let $\breve{A}, \breve{B}, \breve{C} \in ECSM_{m\times n}^P$, then

$$\breve{A} \wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \left(\breve{A} \wedge_P \breve{B}\right) \vee_P \left(\breve{A} \wedge_P \breve{C}\right) = \breve{A}.$$

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \in ECSM_{m\times n}^P$.

Then, L.H.S= $\breve{A} \wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \breve{A} \wedge_P \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \vee_P \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle\right)$.

Since $\breve{B} \subseteq_P \breve{C} \Leftrightarrow \tilde{B}_{ij}^b \le \tilde{C}_{ij}^c$ and $\mu_{ij}^b \le \gamma_{ij}^c$ for all i, j.

By Definition 3.5(a),

$$\breve{A} \wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \breve{A} \wedge_P \left(\left\langle [\max(\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}), \max(\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+})], \max(\mu_{ij}^b, \gamma_{ij}^c) \right\rangle\right) \text{ for all i, j}$$

$$= \breve{A} \wedge_P \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle$$

$$= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle.$$

Since $\breve{A} \subseteq_P \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \le \gamma_{ij}^c$ for all i, j.

By Definition 3.5(b),

$$= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j}$$

$$= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle$$

$$= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$$

$$= \breve{A}.$$

R.H.S= $\left(\breve{A} \wedge_P \breve{B}\right) \vee_P \left(\breve{A} \wedge_P \breve{C}\right)$.

Then, $\left(\breve{A} \wedge_P \breve{B}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right)$.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \le \mu_{ij}^b$ for all i, j.

By Definition 3.5(b),

$$\begin{aligned}\left(\breve{A} \wedge_P \breve{B}\right) &= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle.\end{aligned}$$

Then, $\left(\breve{A} \wedge_P \breve{C}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_P \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle\right)$.

Since $\breve{A} \subseteq_P \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \leq \gamma_{ij}^c$ for all i, j.

Again by Definition 3.5(b),

$$\begin{aligned}\left(\breve{A} \wedge_P \breve{C}\right) &= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \min\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \\ \left(\breve{A} \wedge_P \breve{B}\right) \vee_P \left(\breve{A} \vee_P \breve{C}\right) &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_P \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \\ &= \breve{A} \vee_P \breve{A}.\end{aligned}$$

By Proposition 3.9(i), $\left(\breve{A} \wedge_P \breve{C}\right) \vee_P \left(\breve{B} \wedge_P \breve{C}\right) = \breve{A}$.

Hence, $\breve{A} \wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \left(\breve{A} \wedge_P \breve{B}\right) \vee_P \left(\breve{A} \wedge_P \breve{C}\right) = \breve{A}$.

Theorem 3.24 *Let* $\breve{A}, \breve{B}, \breve{C} \in ECSM_{m\times n}^R$, *then*

$$\breve{A} \wedge_R \left(\breve{B} \vee_R \breve{C}\right) = \left(\breve{A} \wedge_R \breve{B}\right) \vee_R \left(\breve{A} \wedge_R \breve{C}\right) = \breve{A}.$$

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle, \breve{C} = \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \in ECSM_{m\times n}^R$.

Then, L.H.S= $\breve{A} \wedge_R \left(\breve{B} \vee_R \breve{C}\right) = \breve{A} \wedge_R \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \vee_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle\right)$.

Since $\breve{B} \subseteq_R \breve{C} \Leftrightarrow \tilde{B}_{ij}^b \leq \tilde{C}_{ij}^c$ and $\mu_{ij}^b \geq \gamma_{ij}^c$ for all i, j.

By Definition 3.7(a),

$$\begin{aligned}\breve{A} \wedge_R \left(\breve{B} \vee_R \breve{C}\right) &= \breve{A} \wedge_R \left(\left\langle [\max(\tilde{B}_{ij}^{b^-}, \tilde{C}_{ij}^{c^-}), \max(\tilde{B}_{ij}^{b^+}, \tilde{C}_{ij}^{c^+})], \min(\mu_{ij}^b, \gamma_{ij}^c) \right\rangle\right) \text{ for all i, j} \\ &= \breve{A} \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle.\end{aligned}$$

Since $\breve{A} \subseteq_R \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j.

By Definition 3.7(b),

$$\begin{aligned}&= \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j} \\ &= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \\ &= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \\ &= \breve{A}.\end{aligned}$$

R.H.S= $\left(\breve{A} \wedge_R \breve{B}\right) \vee_R \left(\breve{A} \wedge_R \breve{C}\right)$.

Then, $\left(\breve{A} \wedge_R \breve{B}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right)$.

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \geq \mu_{ij}^b$ for all i, j.

By Definition 3.7(b),

$$\left(\breve{A} \wedge_R \breve{B}\right) = \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \gamma_{ij}^a \right\rangle$$
$$= \left\langle \tilde{A}_{ij}^a, \gamma_{ij}^a \right\rangle.$$

Then, $\left(\breve{A} \wedge_R \breve{C}\right) = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{C}_{ij}^{c^-}, \tilde{C}_{ij}^{c^+}], \gamma_{ij}^c \right\rangle\right).$

Since $\breve{A} \subseteq_R \breve{C} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{C}_{ij}^c$ and $\lambda_{ij}^a \geq \gamma_{ij}^c$ for all i, j.

Again by Definition 3.7(b),

$$\left(\breve{A} \wedge_R \breve{C}\right) = \left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{C}_{ij}^{c^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{C}_{ij}^{c^+}\}], \max\{\lambda_{ij}^a, \gamma_{ij}^c\} \right\rangle \text{ for all i, j}$$
$$= \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle$$
$$= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$$
$$\left(\breve{A} \wedge_R \breve{B}\right) \vee_R \left(\breve{A} \vee_R \breve{C}\right) = \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_R \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$$
$$= \breve{A} \vee_R \breve{A}.$$

By Proposition3.10(i), $\left(\breve{A} \wedge_R \breve{C}\right) \vee_R \left(\breve{B} \wedge_R \breve{C}\right) = \breve{A}$.

Hence, $\breve{A} \wedge_R \left(\breve{B} \vee_R \breve{C}\right) = \left(\breve{A} \wedge_R \breve{B}\right) \vee_R \left(\breve{A} \wedge_R \breve{C}\right) = \breve{A}$.

Theorem 3.25 Let $\breve{A}, \breve{B} \in ECSM_{m\times n}^P$. Then

(i) $\left(\breve{A} \vee_P \breve{B}\right)^c = \breve{B}^c \wedge_P \breve{A}^c$.

(ii) $\left(\breve{A} \wedge_P \breve{B}\right)^c = \breve{B}^c \vee_P \breve{A}^c$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle$ and $\breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle$.

(i) L.H.S $= \left(\breve{A} \vee_P \breve{B}\right)^c = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \vee_P \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right)^c$.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \leq \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \leq \mu_{ij}^b$ for all i, j.

By Definition 3.5(a),

$$= \left(\left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle\right)^c \text{ for all i, j}$$
$$= \left\langle [1-\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, 1-\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], 1-\max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle$$
$$= \left\langle [1-\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}, 1-\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}], 1-\max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle$$
$$= \left\langle [\tilde{C}_{ij}^{c^+}, \tilde{C}_{ij}^{c^-})], \gamma_{ij}^c) \right\rangle$$
$$= \breve{C}.$$

R.H.S $= \breve{B}^c \wedge_P \breve{A}^c$

$$= \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right)^c \wedge_P \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle\right)^c$$
$$= \left\langle [1-\tilde{B}_{ij}^{b^-}, 1-\tilde{B}_{ij}^{b^+}], 1-\mu_{ij}^b \right\rangle \wedge_P \left\langle [1-\tilde{A}_{ij}^{a^-}, 1-\tilde{A}_{ij}^{a^+}], 1-\lambda_{ij}^a \right\rangle$$

$$=\left\langle [1-\tilde{B}_{ij}^{b^+},1-\tilde{B}_{ij}^{b^-}],1-\mu_{ij}^b\right\rangle \wedge_P \left\langle [1-\tilde{A}_{ij}^{a^+},1-\tilde{A}_{ij}^{a^-}],1-\lambda_{ij}^a\right\rangle.$$

Since $\breve{B}^c \subseteq_P \breve{A}^c$, by Definition 3.5(b),

$$=\left\langle [\min\{1-\tilde{B}_{ij}^{b^+},1-\tilde{A}_{ij}^{a^+}\},\min\{1-\tilde{B}_{ij}^{b^-},1-\tilde{A}_{ij}^{a^-}\}],\min\{1-\mu_{ij}^b,1-\lambda_{ij}^a\}\right\rangle$$

$$=\left\langle [\tilde{C}_{ij}^{c^+},\tilde{C}_{ij}^{c^-})],\gamma_{ij}^c)\right\rangle$$

$$=\breve{C}.$$

L.H.S = R.H.S

So $\left(\breve{A} \vee_P \breve{B}\right)^c = \breve{B}^c \wedge_P \breve{A}^c$.

(ii) L.H.S $=\left(\breve{A} \wedge_P \breve{B}\right)^c = \left(\left\langle [\tilde{A}_{ij}^{a^-},\tilde{A}_{ij}^{a^+}],\lambda_{ij}^a\right\rangle \wedge_P \left\langle [\tilde{B}_{ij}^{b^-},\tilde{B}_{ij}^{b^+}],\mu_{ij}^b\right\rangle\right)^c$.

Since $\breve{A} \subseteq_P \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \le \mu_{ij}^b$ for all i, j.

By Definition 3.5(b),

$$=\left(\left\langle [\min\{\tilde{A}_{ij}^{a^-},\tilde{B}_{ij}^{b^-}\},\min\{\tilde{A}_{ij}^{a^+},\tilde{B}_{ij}^{b^+}\}],\min\{\lambda_{ij}^a,\mu_{ij}^b\}\right\rangle\right)^c \text{ for all i, j}$$

$$=\left\langle [1-\min\{\tilde{A}_{ij}^{a^-},\tilde{B}_{ij}^{b^-}\},1-\min\{\tilde{A}_{ij}^{a^+},\tilde{B}_{ij}^{b^+}\}],1-\min\{\lambda_{ij}^a,\mu_{ij}^b\}\right\rangle$$

$$=\left\langle [1-\min\{\tilde{A}_{ij}^{a^+},\tilde{B}_{ij}^{b^+}\},1-\min\{\tilde{A}_{ij}^{a^-},\tilde{B}_{ij}^{b^-}\}],1-\min\{\lambda_{ij}^a,\mu_{ij}^b\}\right\rangle$$

$$=\left\langle [\tilde{C}_{ij}^{c^+},\tilde{C}_{ij}^{c^-})],\gamma_{ij}^c)\right\rangle$$

$$=\breve{C}.$$

R.H.S = $\breve{B}^c \vee_P \breve{A}^c$

$$=\left(\left\langle [\tilde{B}_{ij}^{b^-},\tilde{B}_{ij}^{b^+}],\mu_{ij}^b\right\rangle\right)^c \vee_P \left(\left\langle [\tilde{A}_{ij}^{a^-},\tilde{A}_{ij}^{a^+}],\lambda_{ij}^a\right\rangle\right)^c$$

$$=\left\langle [1-\tilde{B}_{ij}^{b^-},1-\tilde{B}_{ij}^{b^+}],1-\mu_{ij}^b\right\rangle \vee_P \left\langle [1-\tilde{A}_{ij}^{a^-},1-\tilde{A}_{ij}^{a^+}],1-\lambda_{ij}^a\right\rangle$$

$$=\left\langle [1-\tilde{B}_{ij}^{b^+},1-\tilde{B}_{ij}^{b^-}],1-\mu_{ij}^b\right\rangle \vee_P \left\langle [1-\tilde{A}_{ij}^{a^+},1-\tilde{A}_{ij}^{a^-}],1-\lambda_{ij}^a\right\rangle.$$

Since $\breve{B}^c \subseteq_P \breve{A}^c$, by Definition 3.5(a),

$$=\left\langle [\max\{1-\tilde{B}_{ij}^{b^+},1-\tilde{A}_{ij}^{a^+}\},\max\{1-\tilde{B}_{ij}^{b^-},1-\tilde{A}_{ij}^{a^-}\}],\max\{1-\mu_{ij}^b,1-\lambda_{ij}^a\}\right\rangle$$

$$=\left\langle [\tilde{C}_{ij}^{c^+},\tilde{C}_{ij}^{c^-})],\gamma_{ij}^c)\right\rangle$$

$$=\breve{C}.$$

L.H.S = R.H.S

So $\left(\breve{A} \wedge_P \breve{B}\right)^c = \breve{B}^c \vee_P \breve{A}^c$.

Theorem 3.26 Let $\breve{A}, \breve{B} \in ECSM_{m\times n}^R$. Then

(i) $\left(\breve{A} \vee_R \breve{B}\right)^c = \breve{B}^c \wedge_R \breve{A}^c$.

(ii) $\left(\breve{A} \wedge_R \breve{B}\right)^c = \breve{B}^c \vee_R \breve{A}^c$.

Proof. Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-},\tilde{A}_{ij}^{a^+}],\lambda_{ij}^a\right\rangle$ and $\breve{B} = \left\langle [\tilde{B}_{ij}^{b^-},\tilde{B}_{ij}^{b^+}],\mu_{ij}^b\right\rangle$.

(i) L.H.S $= \left(\breve{A} \vee_R \breve{B}\right)^c = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \vee_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right)^c.$

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(a),

$$= \left(\left\langle [\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle\right)^c \text{ for all i, j}$$
$$= \left\langle [1-\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, 1-\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], 1-\min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle$$
$$= \left\langle [1-\max\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}, 1-\max\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}], 1-\min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle$$
$$= \left\langle [\tilde{C}_{ij}^{c^+}, \tilde{C}_{ij}^{c^-})], \gamma_{ij}^c) \right\rangle.$$
$$= \breve{C}.$$

R.H.S = $\breve{B}^c \wedge_R \breve{A}^c$

$$= \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right)^c \wedge_R \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle\right)^c$$
$$= \left\langle [1-\tilde{B}_{ij}^{b^-}, 1-\tilde{B}_{ij}^{b^+}], 1-\mu_{ij}^b \right\rangle \wedge_R \left\langle [1-\tilde{A}_{ij}^{a^-}, 1-\tilde{A}_{ij}^{a^+}], 1-\lambda_{ij}^a \right\rangle$$
$$= \left\langle [1-\tilde{B}_{ij}^{b^+}, 1-\tilde{B}_{ij}^{b^-}], 1-\mu_{ij}^b \right\rangle \wedge_R \left\langle [1-\tilde{A}_{ij}^{a^+}, 1-\tilde{A}_{ij}^{a^-}], 1-\lambda_{ij}^a \right\rangle.$$

Since $\breve{B}^c \subseteq_R \breve{A}^c$, by Definition 3.7(b),

$$= \left\langle [\min\{1-\tilde{B}_{ij}^{b^+}, 1-\tilde{A}_{ij}^{a^+}\}, \min\{1-\tilde{B}_{ij}^{b^-}, 1-\tilde{A}_{ij}^{a^-}\}], \max\{1-\mu_{ij}^b, 1-\lambda_{ij}^a\} \right\rangle$$
$$= \left\langle [\tilde{C}_{ij}^{c^+}, \tilde{C}_{ij}^{c^-})], \gamma_{ij}^c) \right\rangle$$
$$= \breve{C}.$$

L.H.S = R.H.S

So $\left(\breve{A} \vee_R \breve{B}\right)^c = \breve{B}^c \wedge_R \breve{A}^c$.

(ii) L.H.S $= \left(\breve{A} \wedge_R \breve{B}\right)^c = \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle \wedge_R \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right)^c.$

Since $\breve{A} \subseteq_R \breve{B} \Leftrightarrow \tilde{A}_{ij}^a \le \tilde{B}_{ij}^b$ and $\lambda_{ij}^a \ge \mu_{ij}^b$ for all i, j.

By Definition 3.7(b),

$$= \left(\left\langle [\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, \min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle\right)^c \text{ for all i, j}$$
$$= \left\langle [1-\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}, 1-\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}], 1-\max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle$$
$$= \left\langle [1-\min\{\tilde{A}_{ij}^{a^+}, \tilde{B}_{ij}^{b^+}\}, 1-\min\{\tilde{A}_{ij}^{a^-}, \tilde{B}_{ij}^{b^-}\}], 1-\max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle$$
$$= \left\langle [\tilde{C}_{ij}^{c^+}, \tilde{C}_{ij}^{c^-})], \gamma_{ij}^c) \right\rangle$$
$$= \breve{C}.$$

R.H.S = $\breve{B}^c \vee_R \breve{A}^c$

$$= \left(\left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle\right)^c \vee_R \left(\left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle\right)^c$$

$$= \left\langle [1-\tilde{B}_{ij}^{b^-}, 1-\tilde{B}_{ij}^{b^+}], 1-\mu_{ij}^{b} \right\rangle \vee_R \left\langle [1-\tilde{A}_{ij}^{a^-}, 1-\tilde{A}_{ij}^{a^+}], 1-\lambda_{ij}^{a} \right\rangle$$

$$= \left\langle [1-\tilde{B}_{ij}^{b^+}, 1-\tilde{B}_{ij}^{b^-}], 1-\mu_{ij}^{b} \right\rangle \vee_R \left\langle [1-\tilde{A}_{ij}^{a^+}, 1-\tilde{A}_{ij}^{a^-}], 1-\lambda_{ij}^{a} \right\rangle.$$

Since $\breve{B}^{c} \subseteq_R \breve{A}^{c}$, by Definition 3.7(a),

$$= \left\langle [\max(1-\tilde{B}_{ij}^{b^+}, 1-\tilde{A}_{ij}^{a^+}), \max(1-\tilde{B}_{ij}^{b^-}, 1-\tilde{A}_{ij}^{a^-})], \min(1-\mu_{ij}^{b}, 1-\lambda_{ij}^{a}) \right\rangle$$

$$= \left\langle [\tilde{C}_{ij}^{c^+}, \tilde{C}_{ij}^{c^-})], \gamma_{ij}^{c}) \right\rangle$$

$$= \breve{C}.$$

L.H.S = R.H.S

So $\left(\breve{A} \wedge_R \breve{B}\right)^c = \breve{B}^{c} \vee_R \breve{A}^{c}$.

References:

1. Chen D., Tsang E.C.C., Yeung D.S., and Wang X.," The parameterization reduction of soft sets and its applications", Computers and Mathematices with Applications 57, 2009, 1547-1553.
2. Chinnaduraia V and Barkavi S., "Cubic soft matrices", International Journal for Research in Mathematics and Statistics, Volume-2, Issue-4, April,2017, 1-22.
3. Jun Y.B., Kim C.S. and Yang K.O., "Cubic sets", Annals of fuzzy Mathematics and Informatics, 4, 2012, 83-98.
4. Kim J.B., "Idempotent and Inverses in Fuzzy Matrices", Malaysian Math, 6(2), 1983, 57-61.
5. Maji P.K., Biswas R. and Roy A.R., "Fuzzy soft sets", Journal of Fuzzy Mathematics, 9, 2001, 589-602.
6. Maji P.K and Roy A.R., "Soft set theory", Computers and Mathematics with Applications 45, 2003, 555-562.
7. Meenakshi A.R., " Fuzzy Matrix Theory and Applications", MJP Publishers, 2008.
8. Molodstov D.A., Soft set theory-first results, Computers and Mathematics with Applications 37, 1999, 19-31.
9. Muhiuddin G., Al-roqi A.M., "Cubic soft sets with applications in BCK/BCL-algebras", Annals of Fuzzy Mathematics and Informatics, In press.
10. Pei D. and Miao D., "From soft sets to information systems", in Proceedings of the IEEE International Conference on Granular Computing, 2, 2005, 617-621.
11. Thomason M.G., " Convergence power of fuzzy matrix", J. of Math. Anal. Appl. 57, 1977, 476-486.
12. Zadeh L.A., "Fuzzy sets", Information and Control 8, 1965, 338-353.