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GENERATING INFINITELY MANY PERFECT SQUARES AMONG NARAYANA NUMBERS

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Abstract:

The Narayana numbers $N(n,k)\ for\ 1 \leq k \leq n$ form a triangular array of positive integers, introduced in 1915-1916 by the combinatorialist P. A. MacMahon, and rediscovered in 1955 by the statistician T. V. Narayana. Among Narayana numbers, it turns out that N (1728,28) is a perfect square. A natural question that would arise is whether there exist other values of a such that $N(a, 28)$ forms a perfect square? In this paper, we discuss ways of determining infinitely many values of a, for given choice of $b > 1$, such that the Narayana numbers $N(a, b)$forms a perfect square.

Key Words: Narayana Numbers, Binomial Coefficients, Pell's Equation, Divisibility Property, Continued Fraction

1. Introduction:

Narayana numbers are a sequence of positive integers that arise in various mathematical contexts, including combinatorics, algebra, and geometry. The Narayana numbers are denoted by $N(n,k), n\epsilon N^+, 1 \leq k \leq n$ and forms a triangular array of natural numbers, called Narayana triangle that occurs in various counting problems, which is a fundamental problem in combinatorial geometry. Solution for Dyck words, a counting problem where number of words containing n pairs of parentheses, which are correctly matched and which contain k distinct nesting can be given in terms of Narayana numbers $N(n,k)$. Narayana numbers have numerous arithmetic properties, which can be proved through corresponding recurrence relations, generating functions, and they possess interesting algebraic identities. R. Sivaraman discussed two interesting results on Catalan numbers [5] and combinatorial identities of binomial coefficients and its properties associated with Narayana numbers [6]. These papers provide good insight in to understanding of Narayana numbers. A perfect square is a non-negative integer that can be expressed as the product of an integer with itself. This paper provides three conditions for which Narayana numbers can also be perfect squares. Detailed proofs have been presented.

2. Definitions:

2.1 Binomial Coefficient:

The numbers formed through two integers n and r where $n \geq 0, 0 \leq r \leq n$ given by the expression $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ are called binomial coefficients. Also, in binomial coefficients $\binom{n}{r} = \binom{n}{n-r} = \frac{n}{r}\binom{n-1}{r-1}$. These numbers got their name because they form the coefficients of the binomial expansion $(a+b)^n$ for $n \geq 0$. Binomial coefficients forms the basis for proving several combinatorial identities.

2.2 Narayana Numbers:

The Narayana numbers are positive integers defined by

$$N(a,b) = \frac{1}{a} \times \binom{a}{b} \times \binom{a}{b-1}, \text{ where } 1 \leq b \leq a+1.$$

2.3 Pell's Equation:

Pell's equation is an equation of the form $x^2 - dy^2 = 1$, where $x$and y are some integers and d is some positive integer.

2.4 Continued Fraction:

An expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\dots\dots}}}}$$

is called a continued fraction. In general, the numbers $a_0, a_1, a_2, a_3 \dots, b_0, b_1, b_2, b_3 \dots$ may be either real numbers or complex numbers, and the number of terms may be finite or infinite.

3. Results on perfect squares on Narayana numbers for the choice of $b > 1$:

3.1 Theorem:

The Narayana numbers $N(a,b)$are perfect squares if and only if $ab(a-b+1) = \alpha^2$ for some integer α.

Proof:

By definition, Narayana numbers are given by

$$N(a,b) = \frac{1}{a} \times \binom{a}{b} \times \binom{a}{b-1}$$
$$= \frac{b}{a} \times \binom{a}{b} \times \frac{1}{b} \times \binom{a}{b-1}$$
$$= \frac{b}{a} \times \binom{a}{b} \times \frac{1}{b} \times \frac{a!}{(b-1)!(a-b+1)!} = \frac{b}{a} \times \binom{a}{b} \times \frac{a!}{b(b-1)!\,(a-b+1)!}$$
$$= \frac{b}{a} \times \binom{a}{b} \times \frac{a!}{b!\,(a-b+1)(a-b)!}$$
$$= \frac{b}{a} \times \binom{a}{b} \times \frac{1}{(a-b+1)} \times \frac{a!}{b!\,(a-b)!}$$
$$= \frac{b}{a} \times \binom{a}{b} \times \frac{1}{(a-b+1)} \times \binom{a}{b}$$
$$= \frac{b}{a(a-b+1)} \times \binom{a}{b}^2 \qquad (1)$$

Further, multiply and divide (1) by b,

$$N(a,b) = \frac{b}{a(a-b+1)} \times \binom{a}{b}^2 \times \frac{b}{b} = \frac{b^2}{ab\,(a-b+1)} \times \binom{a}{b}^2$$

Hence, $N(a,b)$ will be a perfect square for the choice of $b > 1$, if and only if $ab(a-b+1)$is a perfect square (i.e.) $ab(a-b+1) = \alpha^2$ for some integer α.
This completes the proof.

3.2 Theorem 2:

If $b = ds^2$ where d is a square free integer and if $ab(a-b+1) = \alpha^2$ for some integer α, then the solutions of the Pell's equation $n^2 - dm^2 = (b-1)^2$where m is even, are given by

$$(n,m) = \left(2a+1-b, \frac{2s\alpha}{b}\right)$$

Proof:

First we show that $b/s\alpha$.

$$ab(a-b+1) = \alpha^2 \;\Rightarrow\; \frac{\alpha^2}{b} = a(a-b+1) \qquad (2)$$

From (2), we notice that $\frac{\alpha^2}{b}$ is an integer

Choosing $b = ds^2$where d is a square free integer, we have

$\frac{\alpha^2}{b} = \frac{\alpha^2}{ds^2} \Rightarrow s^2/\alpha^2$($\because$ d is square free)

s/α. Hence, $ds/\alpha$and so ds^2 divides $s\alpha$

Therefore,

$$b/s\alpha \qquad (3)$$

Thus, $m = \frac{2s\alpha}{b}$ is an even integer.

We now prove that $(n,m) = \left(2a+1-b, \frac{2s\alpha}{b}\right)$ forms solutions of the Pell's equation $n^2 - dm^2 = (b-1)^2$

$$n^2 - dm^2 = (2a+1-b)^2 - d\left(\frac{2s\alpha}{b}\right)^2$$
$$= (2a+1-b)^2 - d\left(\frac{2s\alpha}{ds^2}\right)^2$$
$$= (2a+1-b)^2 - d\,\frac{4\alpha^2}{d^2s^2}$$
$$= (2a+1-b)^2 - \frac{4\alpha^2}{b}$$
$$= (2a+1-b)^2 - 4(a(a-b+1)) \qquad \text{(from (2))}$$
$$= b^2 - 2b + 1 = (b-1)^2$$

Therefore$(n,m) = \left(2a+1-b, \frac{2s\alpha}{b}\right)$ forms solutions to the Pell's equation$n^2 - dm^2 = (b-1)^2$, where m is an even integer. This completes the proof.

3.3 Theorem 3:

$N(a,b) = c^2$ if and only if $n^2 - dm^2 = (b-1)^2$, for $(n,m) = \left(2a+1-b, \frac{2s\alpha}{b}\right)$ where m is even and $b = ds^2$.

Proof:

From theorem 1, we know that $N(a,b) = c^2$if and only if $ab(a-b+1) = \alpha^2$ for some integer α. But if $ab(a-b+1) = \alpha^2$and if $b = ds^2$then by theorem 2, we know that $(n,m) = \left(2a+1-b, \frac{2s\alpha}{b}\right)$ are solutions to the Pell's equation$n^2 - dm^2 = (b-1)^2$, where m is an even integer. This completes the proof.

From theorem 3, we notice that for the choice of $b > 1$ we need to choose $a = \frac{n+b-1}{2}$ so that $N(a, b) = c^2$.

4. Illustration:

Using theorems 2 and 3, we now present an illustration to determine values of a for given $b > 1$ such that $N(a, b) = c^2$

Let us choose $b = 28$, so that $b = 7 \times 2^2$

From this $d = 7$ & $s = 2$. Thus the Pell's equation is given by $n^2 - 7m^2 = 27^2$ (4)

To solve (4), first we find solutions for $n^2 - 7m^2 = 1$ (5)

In order to do this, first we consider the following continued fraction expansion given by

$$\sqrt{7} = 3 - \cfrac{2}{6 - \cfrac{2}{6 - \cfrac{2}{6 - \cfrac{2}{6 - \cdots}}}} \quad (6)$$

We notice that $c_0 = (1,0)$ is a trivial solution to (5)

To determine the non – trivial solutions of (5), we extract the successive convergents from the continued fraction expansion in (6), to obtain

$$c_1 = (8,3), c_2 = (45,17), c_3 = (127,48), c_4 = (717,271) \ldots$$

Among these, $(n, m) = (127,48)$ satisfies $n^2 - 7m^2 = 1$, such that m is even.

$$27^2 n^2 - 7 \times 27^2 m^2 = 27^2 \Rightarrow (27n)^2 - 7(27m)^2 = 27^2$$

Hence from the solution $(n, m) = (127,48)$to (5), we obtain the solution, $(27n, 27m) = (3429, 1296)$ to (4).

Now from, $n = 2a + 1 - b \Rightarrow 3429 = 2a + 1 - 28 \Rightarrow a = 1728$

Therefore for the choice of $b = 28$, we find that $a = 1728$, such that $N(1728,28)$ will be a perfect square.

The next set of solution for $N(a, b) = c^2$ for the choice of $b = 28$

Choose $c_5 = (n, m) = (32257, 12192)$, $(27n, 27m) = (870939, 329184)$

$$a = \frac{870939 + 28 - 1}{2} = 435483$$

Therefore, for the choice of $b = 28$, we find that $a = 435483$, such that $N(435483,28)$ will be a perfect square.

Similarly by extracting odd order convergents from the continued fraction expansion of (6), we notice that there are infinitely many values can be obtained for the given choice of $b > 1$ such that $N(a, b)$ is a perfect square.

5. Conclusion:

In this paper, we have answered the question of determining values of a, for given values of $b > 1$, such that $N(a, b)$ becomes perfect squares. The three theorems discussed in the paper provides the way towards meeting this objective. A detailed illustration making use of Continued Fraction expansion of $\sqrt{7}$ is provided to generate consecutive solutions to the associated Pell's equation discussed in the paper in theorem 2. In doing so, we notice that for $b = 28$, N (1728,28) and $N(435483,28)$are perfect squares. But this doesn't come under any of three categories discussed in [21]. Thus in this paper, we have provided conditions and derived results towards generating perfect squares among Narayana Numbers when $b > 1$ is given. These results will pave way for extracting as many perfect squares as possible among the famous Narayana Numbers.

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